TIGHT BOUNDS ON THE PROBABILITY OF A UNION WITH APPLICATIONS TO NON-UNIFORM SIGNALING OVER AWGN CHANNELS

by

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Abstract

We establish three tight bounds (two lower bounds and one upper bound) on the probability of a finite union of events. The three bounds - which are expressed in terms of only the individual and pairwise event probabilities - are called the KAT lower bound, the Stepwise lower bound and the Greedy Algorithm upper bound. We then employ these bounds to investigate the error performance of two dimensional non-uniform signaling (with Gray mapping) in the presence of additive white Gaussian noise (AWGN). Both the symbol error rate (SER) and the bit error rate (BER) are considered under the maximum-a-posteriori (MAP) decoding criterion.

The KAT lower bound is derived by a linear algebraic method. We provide theoretical proofs as well as numerical results to show that it is at least as good as two similar lower bounds, one by de Caen [9] and the other by Dawson and Sankoff [8]. The Stepwise lower bound is a suboptimal instance of Kounias inequality [15]; but it has a lower computational complexity. The Greedy Algorithm upper bound is obtained by employing a greedy algorithm which is optimal for constructing a spanning tree with minimum or maximum weight.

We then examine the applications of these bounds to M-ary phase shift keying (PSK) and M-ary quadrature amplitude modulation (QAM) signaling over AWGN channels. Exact simulation results are also provided to show how well these bounds perform. Plots include SER and BER results for 8-PSK, 16-PSK, 32-PSK and 16-QAM with P{X=0}=0.5, 0.7 and 0.9. It is shown from the plots that the bounds agree with the simulation results even for severe channel conditions (i.e. low SNR). In particular, the Stepwise bound and the Greedy Algorithm bound are extremely close to the simulation over a wide range of SNR.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

A fundamental problem of a communication system is to determine, after the signal is corrupted by the channel noise, the probability for the decoder to decode correctly. Equivalently, we wish to know the error probabilities associated with this communication system.

Figure 1.1 illustrates the functional diagram and the basic elements of a digital communication system.

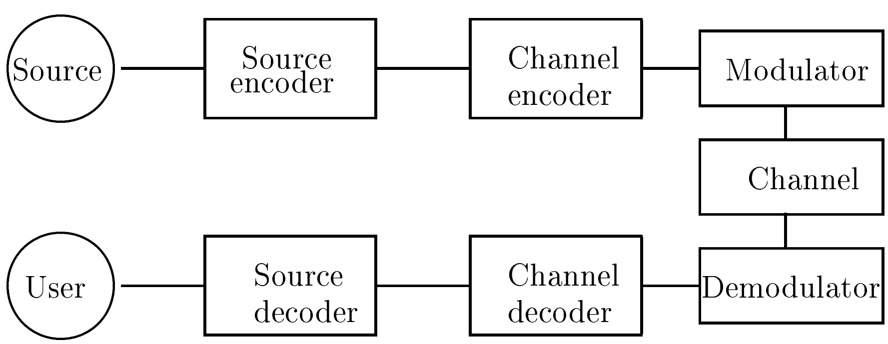

Figure 1.1: Basic elements of a digital communication system

In this thesis, we treat M-ary phase shift keying (PSK) and M-ary quadrature amplitude modulation (QAM) signals in the presence of additive white Gaussian noise (AWGN). The AWGN channel model is considered here because it is mathematically tractable and applies to a broad class of physical communication channels. We apply three tight bounds on the probability of a union to estimate the performance of the system. Both Symbol Error Rate (SER), P_s, and Bit Error Rate (BER), P_b, are considered. In addition, we focus on the maximum-a-posteriori (MAP) decision criterion with non-uniform signaling instead of the Maximum Likelihood (ML) criterion. The plots in Chapter 4 will show the strong performance of the bounds for both SER and BER.

1.1 Literature Review

1.1.1 Bounds on the probability of a union

We begin our work by considering bounds on the probability of a union $P(A_1 \cup \cdots \cup A_N)$, where $A_1, A_2, \ldots, A_N$ form a finite family of events on a given probability space (Ω, P) and N is a fixed positive integer. Without loss of generality, we assume that Ω is finite (for a general probability space, the problem can be directly reduced to the finite case since there are only finitely many Boolean atoms specified by the A_i's). If we have complete information about these events $A_1, A_2, \ldots, A_N$, inclusion and exclusion method [12, p. 1] proved the following formula to compute the exact value of $P(A_1 \cup \cdots \cup A_N)$:

$$\begin{aligned} P\left(\bigcup_{i=1}^{N} A_i\right) &= \sum_{i=1}^{N} P(A_i) - \sum_{i=1}^{N}\sum_{j=1}^{i-1} P(A_i \cap A_j) + \sum_{i=1}^{N}\sum_{j=1}^{i-1}\sum_{k=1}^{j-1} P(A_i \cap A_j \cap A_k) \\ &\quad -... + (-1)^N P(A_1 \cap A_2 \cap \ldots \cap A_N), \end{aligned} \tag{1.1}$$

where the fourth term is the sum of probabilities of intersections of any four, and so on.

The sign of each term depends on the number of events in the intersections. For many real applications, we have no idea about what is going on for higher intersections. Therefore, we cannot use formula (1.1). In particular, it is not practical to apply (1.1) to communication systems. One more event in the intersection often means an additional layer of integration. For a communication system, the number of events is usually huge (e.g. $N = 2^{32}$ if a channel code is employed). How can we handle so many layers of integrations even if we ignore the computation speed and accuracy? Because of this, it is important to establish lower and upper bounds using intersections of a lesser number of events. Over the past years, different kinds of bounds have appeared in the literature. The objective is to find bounds as tight as possible using less information. Next, we will take a quick look at some of the bounds.

In 1967, D. Dawson and D. Sankoff proved the following lower bound [8]:

$$P\left(\bigcup_{i=1}^{N} A_i\right) \geq \frac{\theta S_1^2}{(2-\theta)S_1 + 2S_2} + \frac{(1-\theta)S_1^2}{(1-\theta)S_1 + 2S_2}.$$

where

$$S_1 \triangleq \sum_{i=1}^{N} P(A_i),$$

$$S_2 \triangleq \sum_{i=1}^{N} \sum_{j=1}^{i-1} P(A_i \cap A_j),$$

and

$$\theta \triangleq \frac{2S_2}{S_1} - \left\lfloor \frac{2S_2}{S_1} \right\rfloor.$$

Note that $\theta \in [0, 1)$.

In 1996, D. de Caen provided another lower bound [9]:

$$P\left(\bigcup_{i=1}^{N} A_i\right) \geq \sum_{i=1}^{N} \frac{P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j)}.$$

In [9], de Caen pointed out that the Dawson-Sankoff bound is upper bounded by de Caen bound when $\theta = 0$.

In Chapter 2, we will show that these two bounds are special cases of the KAT lower bound and never stronger than the KAT bound.

The previous three bounds are non-linear. In 1968, Kounias established the following linear lower bound [15]:

$$P(\bigcup_{i=1}^{N} A_i) \geq \max_{\mathcal{J}}\{\sum_{i\in\mathcal{J}} P(A_i) - \sum_{\substack{i,j\in\mathcal{J}\\ i<j}} P(A_i \bigcap A_j)\},$$

where the maximum is taken over all subsets $\mathcal{J}$ of set $\mathcal{S}\stackrel{\triangle}{=}\{1, 2, \ldots, N\}$. Since there are in total $2^N - 1$ non-empty subsets of $\mathcal{S}$, the number of comparisons increases exponentially in terms of N. Therefore, the search complexity is huge for large N. In Chapter 2, we will introduce a Stepwise search algorithm into this problem which results in a suboptimal case of the Kounias bound but with a reduced search complexity.

Kounias (1968) also provided a Bonferroni type upper bound as follows [15]:

$$P\left(\bigcup_{i=1}^{N} A_i\right) \leq \sum_{i=1}^{N} P(A_i) - \max_{j} \sum_{i=1, i\neq j}^{N} P(A_i \cap A_j),$$

where the maximum is taken over $j \in \{1, 2, \ldots, N\}$.

The following inequality - of which the above Kounias upper bound is a special case - is due to Hunter (1976) [12]:

$$P\Big(\bigcup_{i=1}^{N} A_i\Big) \leq \sum_{i=1}^{N} P(A_i) - \sum_{(i,j)\in T_0} P(A_i \bigcap A_j),$$

where T_0 is an arbitrary tree spanning the N indices with pairwise events probabilities as the weights of the edges. It is known that there are N^{N-2} different trees on N nodes [26]. Applying different trees to the above inequality yield different upper bounds. There is at least one tree, called an optimal tree, giving the tightest upper bound. We will apply the Greedy algorithm to construct an optimal tree with largest weight.

All of the bounds we mention in this chapter are in terms of the individual and pairwise event probabilities. They are easy to compute when applied to practical communication systems as opposed to bounds that use higher intersections.

1.1.2 Performance bounds on SER and BER

Many papers have appeared in the literature in past years about the symbol error rate (SER) P_s and bit error rate (BER) P_b of M-PSK and QAM modulated signals over AWGN channels [13, 14, 19, 21]. In [21, pp. 269-282], J. Proakis derived the exact formulas for SER and an approximate estimate of BER for uniform M-PSK and QAM signaling with Gray coding over AWGN channels. In 1991, B. Hughes [13] proposed a new upper bound on the SER, which is easy to calculate and more importantly can be applied to any signal set (especially, when the exact formula is unavailable). In addition, it is always stronger than the union and minimum distance bounds. In 1991, M. I. Irshid and I. S. Salous [14] presented a simple technique for evaluating the BER for uniform M-ary PSK signals with Gray, natural binary and folded binary bit-mappings. This technique avoids numerical evaluation of complicated integrals in the direct method and uses instead closed form expressions in terms of error functions.

However, all of the previous work was done under the ML decoding criterion. There has not been much effort for the MAP decision criterion which is optimal and more general. The lack of such results is due to the complex nature of decision regions. (If $\mathbf{I_u}$ is the decision region of $\mathbf{s_u}$, then in $\mathbf{I_u}$, signal $\mathbf{s_u}$ has a higher decision metric than any other signals. For ML, the decision region is usually symmetrical; for MAP, it is always non-symmetric). For instance, the expression for SER is $P_s = \sum_{u=1}^{M} P(\epsilon|\mathbf{s_u})P(\mathbf{s_u})$, where M is the number of signals and ϵ is the event that other signals have higher decision metric than $\mathbf{s_u}$; i.e., it is the event of decoding error. Since the term $P(\epsilon|\mathbf{s_u})$ depends on the decision region $\mathbf{I_u}$, the calculation of this term is therefore complex. For this reason, a number of upper and lower bounds have been derived to bound P_s. Not surprisingly, most of these bounds make use of geometric methods that are valid only for ML decoding and symmetric decision regions, and may be good only in some specific cases, but bad for other cases. For BER, a worse situation occurs. The bounds we provide in this work are based on general bounds for the probability of a union of events, and thus provide a general methodology for error performance analysis.

In previous related work [7], Craig provided an elegant method for computing P_s for two-dimensional M-ary signaling over AWGN channels. Although he implemented his technique exclusively for uniformly distributed (equally likely) signaling used with ML decoding, he briefly considered the problem of non-uniform signaling and gave an expression for P_s (cf equations (13) and (14) in [7]). However, this expression, which is written as a sum of single integrals, explicitly depends on the geometry of the polygonal decision region of each non-uniform signal. In the case of uniform signaling, these regions are straightforward to determine and are symmetrically shaped. But for non-uniform signaling, these regions are non-symmetric and are shaped differently for each signal. Craig's other important contribution is that he derived a new and simple integral expression for the Gaussian tail probability function $Q(x)$. This new expression is of interest since the range of the integration is finite and the integrand is an exponential function.

In [4, 10, 25], Craig's method was extensively generalized to provide a unified framework for the determination of P_s for various uniform modulation schemes over generalized fading channels. In other work, the computation of P_b for uniform M-ary PSK and QAM signaling over AWGN channels was investigated [14, 19]. In an approach similar in spirit to our work [23], G. E. Séguin employed de Caen lower bound to investigate the probability of error for M-ary signals coded with a binary linear code which is corrupted by AWGN with a ML decoder. The result is comparable with Shannon's lower bound. Since de Caen lower bound is weaker than the KAT lower bound, the results in [23] can be tightened by using the new bounds derived in this thesis.

1.2 Contribution

The contributions of this thesis have significance in the fields of both mathematics and engineering. We have the following new results:

- We establish a new lower bound (the KAT lower bound which is named after its three authors [16]: Kuai, Alajaji and Takahara) on the probability of a union of a finite family of events.

- We provide theoretical proofs to show that the KAT bound is always at least as good as two previous lower bounds, one by de Caen and the other by Dawson and Sankoff, and provide numerical examples to illustrate these facts.

- We present an algorithmic lower bound on the probability of a finite union. This bound is a suboptimal instance of the Kounias bound [15] but has less complexity. The performance of this bound is shown to be excellent for our applications.

- We present an algorithmic upper bound on the probability of a finite union. This upper bound - which is expressed in terms of a weighted connected graph G and its spanning tree - is based on a greedy algorithm which constructs the optimal spanning tree in G.

In addition, we examine the application of these bounds to study the SER and the BER when a non-uniform Bernoulli source $\{X_i\}$ is sent using M-ary PSK and

QAM modulations with Gray mapping over AWGN channels and decoded via MAP soft-decision detection.

- We obtain expressions for the SER and BER in terms of unions of finite families of events, and derive closed-form expressions for these events and their pairwise intersections.

- We provide simulation results for both M-PSK and 16-QAM in the cases when $P\{X = 0\}$ is 0.5, 0.7 and 0.9.

- We compare the application of the above bounds with simulation results and the union upper bound in the SER case. All of the bounds agree with the simulation results even during very severe channel conditions (i.e. low SNR).

- We compare the application of the algorithmic bounds with the simulation results in the BER case. Again, the agreement here is excellent.

- To increase precision and efficiency, we adopt a good subroutine in Fortran by Donnely [11] to compute the bivariate normal probabilities.

- A number of plots are attached for M-PSK and 16-QAM for E_b/N_o from -4dB up to 11dB.

1.3 Thesis overview

This thesis is organized in the following manner.

Chapter 2 treats a number of new bounds on the probability of a union of a finite family of events. More specifically, the KAT lower bound is derived by using linear algebraic methods. Two algorithmic bounds are then derived, one lower bound and one upper bound. In addition, numerical comparisons of the KAT bound with two previous lower bounds are presented and theoretical proofs are given for the superiority of the KAT bound over other bounds.

Chapter 3 is devoted to the application of these bounds. We consider a non-uniform i.i.d binary source that is transmitted over an AWGN channel. The modulation techniques are M-PSK and M-QAM with Gray mapping and MAP soft-decision decoding is employed.

Numerical results and discussion are given in Chapter 4. The plots include SER and BER for 8-PSK, 16-PSK, 32-PSK, and 16-QAM with bit input distribution $P\{X = 0\} = 0.5$, 0.7 and 0.9 for E_b/N_o from -4dB to 11dB.

In Chapter 5, we focus on the summary of our work and present future research directions.

The thesis concludes with an appendix. It presents the necessary background about PSK and QAM modulation techniques.

Chapter 2

Bounds on the probability of a union

In this chapter, we present several new bounds - lower and upper bounds - on the probability of a union.

2.1 Lower Bounds

2.1.1 The KAT Lower Bound

Consider a finite family of events $A_1, A_2, \ldots, A_N$ in a finite probability space (Ω, P), where N is a fixed positive integer. For each $x \in \Omega$, let $p(x) \stackrel{\triangle}{=} P(\{x\})$, and let the degree of x – denoted by $\deg(x)$ – be the number of A_i's that contain x. Define

$$B_i(k) \stackrel{\triangle}{=} \{x \in A_i : \deg(x) = k\}$$

and

$$a_i(k) \stackrel{\triangle}{=} P(B_i(k)),$$

where $i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, N$.

Lemma 2.1

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N}\sum_{k=1}^{N}\frac{a_i(k)}{k}.$$

Proof: We know from [9] that

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N}\sum_{x\in A_i}\frac{p(x)}{\deg(x)}.$$

But

$$\begin{aligned}\sum_{x\in A_i}\frac{p(x)}{\deg(x)} &= \sum_{k=1}^{N}\sum_{x\in A_i:\deg(x)=k}\frac{p(x)}{\deg(x)}\\ &= \sum_{k=1}^{N}\sum_{x\in A_i:\deg(x)=k}\frac{p(x)}{k}\\ &= \sum_{k=1}^{N}\frac{1}{k}\sum_{x\in B_i(k)}p(x) = \sum_{k=1}^{N}\frac{a_i(k)}{k}.\end{aligned}$$

This completes the proof. □

Theorem 2.2 KAT Lower bound *[16]*

$$\begin{aligned}P\left(\bigcup_{i=1}^{N} A_i\right) \quad\geq\quad & \sum_{i=1}^{N}\left(\frac{\theta_i P(A_i)^2}{\sum_{j=1}^{N}P(A_i\cap A_j)+(1-\theta_i)P(A_i)}\right.\\ &\quad + \left.\frac{(1-\theta_i)P(A_i)^2}{\sum_{j=1}^{N}P(A_i\cap A_j)-\theta_i P(A_i)}\right), \qquad (2.1)\end{aligned}$$

where

$$\theta_i \triangleq \frac{\beta_i}{\alpha_i} - \left\lfloor\frac{\beta_i}{\alpha_i}\right\rfloor \in [0,1),$$

$$\alpha_i \triangleq \sum_{k=1}^{N} a_i(k) = P(A_i)$$

and

$$\beta_i \triangleq \sum_{k=1}^{N}(k-1)a_i(k) = \sum_{j:j\neq i} P(A_i\cap A_j).$$

Proof: From Lemma 2.1, we can write

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N}\sum_{k=1}^{N}\frac{a_i(k)}{k} = \sum_{i=1}^{N} V_i,$$

where

$$V_i \stackrel{\triangle}{=} \sum_{k=1}^{N}\frac{a_i(k)}{k}.$$

To obtain a lower bound on $P\left(\bigcup_{i=1}^{N} A_i\right)$, we proceed by finding (for each i) the minimum of the linear expression

$$V_i = \sum_{k=1}^{N}\frac{a_i(k)}{k} \tag{2.2}$$

subject to the constraints:

$$a_i(k) \geq 0, \quad k = 1, \ldots, N, \tag{2.3}$$

$$\sum_{k=1}^{N} a_i(k) = P(A_i) \stackrel{\triangle}{=} \alpha_i, \tag{2.4}$$

and

$$\sum_{k=1}^{N}(k-1)a_i(k) = \sum_{j:j\neq i} P(A_i \cap A_j) \stackrel{\triangle}{=} \beta_i. \tag{2.5}$$

This constrained minimization problem is solved using the same methodology as proposed in [8].

Step 1: For $r \geq 2$, solving (2.4) for $a_i(r-1)$ gives:

$$a_i(r-1) = \alpha_i - \sum_{k:k\neq r-1} a_i(k).$$

Substituting the above expression of $a_i(r-1)$ in (2.5) yields:

$$(r-2)\left[\alpha_i - \sum_{k:k\neq r-1} a_i(k)\right] + \sum_{k:k\neq r-1}(k-1)a_i(k) = \beta_i$$

or

$$\sum_{k:k\neq r-1}[k-(r-1)]a_i(k) = \beta_i - (r-2)\alpha_i.$$

Dividing by r, we get:

$$\frac{1}{r}\sum_{k:k\neq r-1}[k-(r-1)]a_i(k) = \frac{1}{r}[\beta_i - (r-2)\alpha_i] \tag{2.6}$$

Step 2: Solving (2.5) for $a_i(r)$ gives:

$$a_i(r) = \frac{1}{r-1}\left[\beta_i - \sum_{k:k\neq r}(k-1)a_i(k)\right].$$

Substituting the expression for $a_i(r)$ in (2.4) yields:

$$\frac{1}{r-1}\left[\beta_i - \sum_{k:k\neq r}(k-1)a_i(k)\right] + \sum_{k:k\neq r}a_i(k) = \alpha_i$$

or

$$\frac{1}{r-1}\sum_{k=1}^{N}(r-k)a_i(k) = \alpha_i - \frac{\beta_i}{r-1}. \tag{2.7}$$

Step 3: Solving (2.6) for $a_i(r)$ and solving (2.7) for $a_i(r-1)$ respectively yield:

$$\frac{a_i(r)}{r} = \frac{\beta_i}{r} - \frac{(r-2)\alpha_i}{r} - \sum_{k:k\neq r}\frac{k-(r-1)}{r}a_i(k)$$

and

$$\frac{a_i(r-1)}{r-1} = \alpha_i - \frac{\beta_i}{r-1} - \sum_{k:k\neq r-1}\frac{r-k}{r-1}a_i(k).$$

Substituting the above two expressions in (2.2) yields:

$$V_i - \frac{\beta_i}{r} + \frac{r-2}{r}\alpha_i + \sum_{k:k\neq r}\frac{k-(r-1)}{r}a_i(k) - \alpha_i + \frac{\beta_i}{r-1} + \sum_{k:k\neq r-1}\frac{r-k}{r-1}a_i(k)$$

$$= \sum_{k:k\neq r-1,r}\frac{a_i(k)}{k},$$

or

$$V_i - \frac{2}{r}\alpha_i + \frac{1}{r(r-1)}\beta_i = \sum_{k=1}^{N}\frac{(r-k)(r-k-1)}{r(r-1)}\frac{a_i(k)}{k} \geq 0,$$

where $r \geq 2$.

Step 4: Define

$$f_i(r) \stackrel{\triangle}{=} \frac{2}{r}\alpha_i - \frac{\beta_i}{r(r-1)}. \tag{2.8}$$

We thus get that

$$V_i \geq f_i(r) \tag{2.9}$$

where

$$r \geq 2.$$

We want to maximize $f_i(r)$ over $r \geq 2$ in order to render (2.9) as tight as possible. Setting

$$\begin{cases} f_i(r) - f_i(r-1) \geq 0 \\ f_i(r) - f_i(r+1) \geq 0 \end{cases},$$

we get

$$1 + \frac{\beta_i}{\alpha_i} \leq r \leq 2 + \frac{\beta_i}{\alpha_i}.$$

Since r is an integer, we obtain

$$1 + \left\lfloor \frac{\beta_i}{\alpha_i} \right\rfloor \leq r \leq 2 + \left\lfloor \frac{\beta_i}{\alpha_i} \right\rfloor. \tag{2.10}$$

Let $r_1' \stackrel{\triangle}{=} 1 + \lfloor \frac{\beta_i}{\alpha_i} \rfloor$, $r_2' \stackrel{\triangle}{=} 2 + \lfloor \frac{\beta_i}{\alpha_i} \rfloor$ and $\theta_i = \frac{\beta_i}{\alpha_i} - \lfloor \frac{\beta_i}{\alpha_i} \rfloor$. So

$$\begin{aligned} f_i(r_1') &= \frac{(1+\theta_i){\alpha_i}^2}{\beta_i + (1-\theta_i)\alpha_i} - \frac{\theta_i {\alpha_i}^2}{\beta_i - \theta_i \alpha_i}, \\ f_i(r_2') &= \frac{\theta_i {\alpha_i}^2}{\beta_i + (2-\theta_i)\alpha_i} + \frac{(1-\theta_i){\alpha_i}^2}{\beta_i + (1-\theta_i)\alpha_i}. \end{aligned}$$

If r_1' is valid – i.e., if $r_1' \geq 2$ – it is easy to prove that $f_i(r_1') \leq f_i(r_2')$. This is verified as follows:

$$\begin{aligned} f_i(r_2') - f_i(r_1') &= \frac{\theta_i {\alpha_i}^2}{\beta_i + (2-\theta_i)\alpha_i} + \frac{(1-\theta_i){\alpha_i}^2}{\beta_i + (1-\theta_i)\alpha_i} \\ &\quad - \frac{(1+\theta_i){\alpha_i}^2}{\beta_i + (1-\theta_i)\alpha_i} + \frac{\theta_i {\alpha_i}^2}{\beta_i - \theta_i \alpha_i} \end{aligned}$$

$$
\begin{aligned}
&= \frac{2\theta_i(\alpha_i)^4}{[\beta_i + (2-\theta_i)\alpha_i][\beta_i + (1-\theta_i)\alpha_i][\beta_i - \theta_i\alpha_i]} \\
&\geq 0.
\end{aligned}
$$

Substituting $f_i(r_2')$ into (2.9) and summing over i yields (2.1). □

Remark: There is another geometric proof for the KAT bound using linear programming. The alternate proof is provided next.

Second Proof of the KAT Bound:

Let $y_k = a_i(k)$, $\alpha = \alpha_i$, $\beta = \beta_i$, where

$$a_i(k) \stackrel{\triangle}{=} P(B_i(k)),$$

$$B_i(k) \stackrel{\triangle}{=} \{x \in A_i : \deg(x) = k\},$$

$$\alpha_i \stackrel{\triangle}{=} \sum_{k=1}^{N} a_i(k) = P(A_i),$$

and

$$\beta_i \stackrel{\triangle}{=} \sum_{k=1}^{N} (k-1)a_i(k) = \sum_{j:j\neq i} P(A_i \cap A_j).$$

as before, $i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, N$.

In (2.2), we wish to

$$\text{Minimize } \sum_{k=1}^{N} y_k/k$$

subject to

$$\sum_{k=1}^{N} y_k = \alpha,$$

$$\sum_{k=1}^{N} (k-1)y_k = \beta,$$

$$y_k \geq 0.$$

Equivalently (cf the dual problem [20] pp 84-110), we wish to

$$\text{Maximize } \alpha x_1 + \beta x_2$$

subject to

$$x_1 + (k-1)x_2 \leq 1/k,$$
$$k = 1, 2, \ldots, N.$$

Therefore, there are N constraints for the dual problem. The set of (x_1, x_2) satisfying the inequality constraints is a polygon in the plane bounded by the lines L_k with equations $x_1 + (k-1)x_2 = 1/k$.

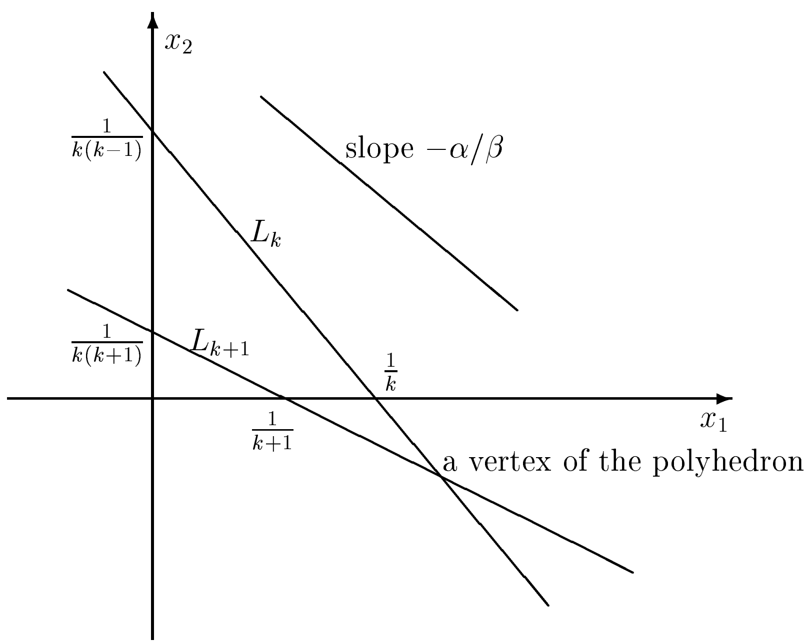

Figure 2.1: Illustration of extreme point solution

For $\alpha x_1 + \beta x_2 = C$, it has a constant slope $-\alpha/\beta$ and is maximized at that vertex of the polyhedron when the level line $\alpha x_1 + \beta x_2 = C$ just touches the polyhedron [20, chapter 2]. That will occur when the slope $-\alpha/\beta$ of the level line lies between the slopes of the two edges of the polyhedron at the vertex; i.e., when

$$\text{slope of } L_k \leq -\alpha/\beta \leq \text{ slope of } L_{k+1},$$

that is,

$$-\frac{1}{k-1} \leq -\alpha/\beta \leq -\frac{1}{k}$$
$$k-1 \leq \beta/\alpha \leq k$$

or

$$\left\lfloor \frac{\beta}{\alpha} \right\rfloor \leq k \leq \left\lfloor \frac{\beta}{\alpha} \right\rfloor + 1.$$

where k corresponds to $r-1$ in (2.10). The rest of the proof is the same as the last one. □

2.1.2 Comparison with de Caen's bound

In a recent paper [9], de Caen presented a lower bound on $P(\cup_{i=1}^{N} A_i)$ in terms of the $P(A_i)$'s and the $P(A_i \cap A_j)$'s.

Lemma 2.3 (de Caen [9]) Let $A_1, A_2, \ldots, A_N$ be any finite family of events in a probability space (Ω, P). Then

$$P\left(\bigcup_{i=1}^{N} A_i\right) \geq \sum_{i=1}^{N} \frac{P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j)}. \tag{2.11}$$

We next demonstrate that our new bound is *always* at least as good as de Caen's bound. More specifically, we prove the following.

Lemma 2.4 Let $A_1, A_2, \ldots, A_N$ be any finite family of events in a probability space (Ω, P). Then

$$\begin{aligned}\sum_{i=1}^{N}\left(\frac{\theta_i P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j) + (1-\theta_i)P(A_i)} \right. &+ \left. \frac{(1-\theta_i)P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j) - \theta_i P(A_i)}\right) \\ &\geq \sum_{i=1}^{N} \frac{P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j)},\end{aligned}$$

where

$$\theta_i \stackrel{\triangle}{=} \frac{\beta_i}{\alpha_i} - \left\lfloor \frac{\beta_i}{\alpha_i} \right\rfloor.$$

In order to prove Lemma 2.4, we need the following fact.

Lemma 2.5 Suppose $a > 0$, $b \geq 0$, and $0 \leq x \leq 1$, then

$$\frac{xa^2}{b+(2-x)a} + \frac{(1-x)a^2}{b+(1-x)a} \geq \frac{a^2}{b+a}.$$

Proof of Lemma 2.5: Let

$$f(x) = \frac{a^2 x}{b+(2-x)a} + \frac{a^2(1-x)}{b+(1-x)a}.$$

- For $b = 0$,

$$f(x) = \frac{a^2 x}{(2-x)a} + a \geq \frac{a^2}{b+a} = a.$$

  We are done.

- For $b > 0$, $f(x)$ is continuous for all $x \in [0,1]$.

$$f'(x) = \frac{a^2 b + 2a^3}{[b+(2-x)a]^2} - \frac{a^2 b}{[b+(1-x)a]^2}.$$

  Let $x_0 \in [0,1]$ such that $f'(x_0) = 0$. Then we get a unique solution

$$x_0 = \frac{2a + b - \sqrt{2ab + b^2}}{2a} \in [0,1],$$

  and

$$\begin{aligned} f(x_0) &= \frac{x_0 a^2}{b+(2-x_0)a} + \frac{(1-x_0)a^2}{b+(1-x_0)a} \\ &= 2a + 2b - 2\sqrt{2ab+b^2}. \end{aligned}$$

  It is easy to prove that

$$2a + 2b - 2\sqrt{2ab+b^2} > \frac{a^2}{b+a}.$$

  Therefore

$$\begin{aligned} \min_{x\in[0,1]} f(x) &= \min\{f(0), f(1), f(x_0)\} \\ &= \min\left\{\frac{a^2}{b+a}, 2a+2b-2\sqrt{2ab+b^2}\right\} = \frac{a^2}{a+b}; \end{aligned}$$

thus,

$$\frac{xa^2}{b+(2-x)a} + \frac{(1-x)a^2}{b+(1-x)a} \geq \frac{a^2}{b+a}$$

for all $x \in [0,1]$. $\square$

Proof of Lemma 2.4: Letting

$$a = P(A_i),\ b = \sum_{j:j\neq i} P(A_i \cap A_j),\ x = \theta_i = \frac{b}{a} - \left\lfloor \frac{b}{a} \right\rfloor$$

in Lemma 2.5 gives

$$\frac{\theta_i P(A_i)^2}{\sum_{j:j\neq i} P(A_i \cap A_j) + (2-\theta_i)P(A_i)} + \frac{(1-\theta_i)P(A_i)^2}{\sum_{j:j\neq i} P(A_i \cap A_j) + (1-\theta_i)P(A_i)}$$
$$\geq \frac{P(A_i)^2}{\sum_{j:j\neq i} P(A_i \cap A_j) + P(A_i)} = \frac{P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j)}.$$

Therefore, (2.1) is always stronger than (2.11); i.e.,

$$\sum_{i=1}^{N} \left(\frac{\theta_i P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j) + (1-\theta_i)P(A_i)} + \frac{(1-\theta_i)P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j) - \theta_i P(A_i)} \right)$$
$$\geq \sum_{i=1}^{N} \frac{P(A_i)^2}{\sum_{j:j\neq i} P(A_i \cap A_j) + P(A_i)}.$$

Note: de Caen's bound is tight (i.e., (2.11) is an equality) if and only if the degrees $\deg(x)$ are constant on each A_i [9]. Since (2.1) is stronger than (2.11), we conclude that the above condition is only a sufficient (but *not* necessary, cf. Example 1 in Section 2.1.4) condition for the tightness of (2.1). $\square$

Lemma 2.6 If $\theta_i = 0$ for all i, then the KAT bound reduces to de Caen's lower bound.

Proof: When $\theta_i = 0$ for all i, substitute $\theta_i = 0$ into the (2.1), then

$$\sum_{i=1}^{N} \left(\frac{\theta_i P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j) + (1-\theta_i)P(A_i)} + \frac{(1-\theta_i)P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j) - \theta_i P(A_i)} \right)$$
$$= \sum_{i=1}^{N} \frac{P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j)}.$$

which is de Caen's bound.

This leads us to conclude that de Caen lower bound is a special case of the KAT lower bound. $\square$

2.1.3 Comparison with the Dawson-Sankoff bound

We prove in this section that our bound is *always* at least as good as the Dawson-Sankoff bound [8, 12].

Lemma 2.7 (Dawson-Sankoff [8]) Let $A_1, A_2, \ldots, A_N$ be any finite family of events in a probability space (Ω, P). Then

$$P\left(\bigcup_{i=1}^{N} A_i\right) \geq \frac{\theta S_1^2}{(2-\theta)S_1 + 2S_2} + \frac{(1-\theta)S_1^2}{(1-\theta)S_1 + 2S_2}, \tag{2.12}$$

where

$$S_1 \triangleq \sum_{i=1}^{N} P(A_i),$$

$$S_2 \triangleq \sum_{i=1}^{N}\sum_{j=1}^{i-1} P(A_i \cap A_j),$$

and

$$\theta \triangleq \frac{2S_2}{S_1} - \left\lfloor \frac{2S_2}{S_1} \right\rfloor.$$

Lemma 2.8 Let $A_1, A_2, \ldots, A_N$ be any finite family of events in a probability space (Ω, P). Then (2.1) is always sharper than (2.12); i.e.,

$$\sum_{i=1}^{N}\left(\frac{\theta_i P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j) + (1-\theta_i)P(A_i)} + \frac{(1-\theta_i)P(A_i)^2}{\sum_{j=1}^{N} P(A_i \cap A_j) - \theta_i P(A_i)}\right)$$
$$\geq \frac{\theta S_1^2}{(2-\theta)S_1 + 2S_2} + \frac{(1-\theta)S_1^2}{(1-\theta)S_1 + 2S_2}.$$

Proof: From the proof of Theorem 2.2, we know that

$$f_i\left(2 + \left\lfloor \frac{\beta_i}{\alpha_i} \right\rfloor\right) \geq f_i(r), \text{for all } r \geq 2,$$

where the function $f_i(\cdot)$ is described in (2.8). In particular, we have that

$$f_i\left(2+\left\lfloor\frac{\beta_i}{\alpha_i}\right\rfloor\right) \geq f_i\left(2+\left\lfloor\frac{\beta}{S_1}\right\rfloor\right),$$

where

$$\beta \stackrel{\triangle}{=} \sum_{i=1}^{N}\sum_{j:j\neq i} P(A_i \cap A_j) = \sum_{i=1}^{N}\beta_i,$$

and

$$S_1 \stackrel{\triangle}{=} \sum_{i=1}^{N}\alpha_i.$$

It can be easily verified that $\beta = 2S_2$, where S_2 is defined in Lemma 2.7.

Noting that $\sum_i f_i\left(2+\lfloor\frac{\beta_i}{\alpha_i}\rfloor\right)$ yields our bound (the right-hand side of (2.1)), and letting $s = 2+\lfloor\frac{\beta}{S_1}\rfloor$ we have

$$\begin{aligned}
\sum_{i=1}^{N} f_i\left(2+\left\lfloor\frac{\beta_i}{\alpha_i}\right\rfloor\right) &\geq \sum_{i=1}^{N} f_i\left(2+\left\lfloor\frac{\beta}{S_1}\right\rfloor\right)\\
&= \frac{2}{s}\sum_{i=1}^{N}\alpha_i - \frac{1}{s(s-1)}\sum_{i=1}^{N}\beta_i\\
&= \frac{2S_1}{s} - \frac{1}{s(s-1)}\beta\\
&= \frac{2S_1}{s} - \frac{2S_2}{s(s-1)}. \qquad (2.13)
\end{aligned}$$

The proof is completed by observing that the right-hand side of (2.13) is indeed equal to the Dawson-Sankoff bound given in (2.12). □

Lemma 2.9 If $\frac{\beta_i}{\alpha_i} = C$ for all i, where C is a constant, then $\theta_i = \theta$ for all i and the KAT lower bound reduces to the Dawson-Sankoff lower bound.

Proof: If $\frac{\beta_i}{\alpha_i} = C$, then $\beta_i = C\alpha_i$. $2S_2 = \sum_{i=1}^{N} C\alpha_i = CS_1$. That is,

$$\frac{\beta_i}{\alpha_i} = \frac{2S_2}{S_1} = C,$$

and

$$\left\lfloor\frac{\beta_i}{\alpha_i}\right\rfloor = \left\lfloor\frac{2S_2}{S_1}\right\rfloor.$$

Therefore, equality holds for the first step in (2.13). Thus, the Dawson-Sankoff lower bound is a special case of the KAT lower bound. □

2.1.4 Numerical examples

Example 1 We first give an example in which our proposed bound is tight. Let $3|n$ (n is a multiple of 3) and

$$A_i = \begin{cases} \{\frac{3i-1}{2}, \frac{3i+1}{2}\}, & \text{if } i \text{ is odd,} \\ \{\frac{3i}{2}-1, \frac{3i}{2}\}, & \text{if } i \text{ is even,} \end{cases}$$

where $1 \leq i \leq \frac{2n}{3}$. Then $A_i \cap A_j \neq \phi$ if and only if $\lceil \frac{i}{2} \rceil = \lceil \frac{j}{2} \rceil$. If the points are uniformly distributed with probability $\frac{1}{n}$, then

$$P(A_i) = \frac{2}{n},$$

$$\sum_{j:j\neq i} P(A_i \cap A_j) = \sum_{j\neq i:\lceil \frac{i}{2} \rceil = \lceil \frac{j}{2} \rceil} P(A_i \cap A_j) = \frac{1}{n},$$

and

$$\theta_i = \frac{1}{2}.$$

Clearly

$$P\left(\bigcup_{i=1}^{\frac{2n}{3}} A_i\right) = 1.$$

(2.1) gives

$$\sum_{i=1}^{\frac{2n}{3}} \left(\frac{\frac{1}{2}(\frac{2}{n})^2}{\frac{3}{n}+\frac{1}{2}\frac{2}{n}} + \frac{\frac{1}{2}(\frac{2}{n})^2}{\frac{3}{n}-\frac{1}{2}\frac{2}{n}} \right) = \sum_{i=1}^{\frac{2n}{3}} \frac{3}{2n} = 1.$$

However (2.11) gives

$$\sum_{i=1}^{\frac{2n}{3}} \frac{(\frac{2}{n})^2}{\frac{3}{n}} = \sum_{i=1}^{\frac{2n}{3}} \frac{4}{3n} = \frac{8}{9}.$$

Thus, in this case, (2.1) is stronger than (2.11).

Example 2 We next consider several systems and compare our bound to the de Caen and Dawson-Sankoff bounds. The different systems are described in Tables 2.1-2.4. The lower bounds for each system are computed below.

System	$P(\cup_i A_i)$	de Caen (2.11)	Dawson (2.12)	KAT (2.1)
I	0.7890	0.7087	0.7007	0.7247
II	0.6740	0.6154	0.6150	0.6227
III	0.7890	0.7048	0.6933	0.7222
IV	0.9689	0.8759	0.8881	0.8911

It can be clearly observed from the above table that the new bound is sharper than the de Caen and the Dawson-Sankoff bounds.

Outcomes x	$p(x)$	A_1	A_2	A_3	A_4	A_5	A_6
x_0	0.012	$\times$		$\times$		$\times$	
x_1	0.022		$\times$		$\times$		$\times$
x_2	0.023	$\times$		$\times$		$\times$	
x_3	0.033		$\times$				
x_4	0.034	$\times$				$\times$	$\times$
x_5	0.044		$\times$	$\times$		$\times$	
x_6	0.045		$\times$			$\times$	$\times$
x_7	0.055		$\times$	$\times$	$\times$		$\times$
x_8	0.056	$\times$		$\times$			
x_9	0.066				$\times$	$\times$	
x_{10}	0.067		$\times$		$\times$	$\times$	
x_{11}	0.077		$\times$		$\times$		
x_{12}	0.078	$\times$			$\times$		$\times$
x_{13}	0.088		$\times$				
x_{14}	0.089	$\times$		$\times$		$\times$	$\times$

Table 2.1: Description of System I with $N = 6$ and $|\cup_{i=1}^{N} A_i| = 15$. An $\times$ in the (i, j)'th entry indicates that outcome $x_i \in A_j$.

Outcomes x	$p(x)$	A_1	A_2	A_3	A_4	A_5	A_6
x_0	0.023	$\times$		$\times$		$\times$	
x_1	0.034		$\times$		$\times$		
x_2	0.045	$\times$		$\times$		$\times$	
x_3	0.056		$\times$				
x_4	0.067	$\times$				$\times$	$\times$
x_5	0.078		$\times$	$\times$		$\times$	
x_6	0.067		$\times$			$\times$	$\times$
x_7	0.056			$\times$	$\times$		$\times$
x_8	0.045	$\times$		$\times$			
x_9	0.038				$\times$	$\times$	
x_{10}	0.011		$\times$		$\times$	$\times$	
x_{11}	0.022		$\times$				
x_{12}	0.033	$\times$			$\times$		$\times$
x_{13}	0.044		$\times$				
x_{14}	0.055	$\times$		$\times$		$\times$	$\times$

Table 2.2: Description of System II with $N = 6$ and $|\cup_{i=1}^{N} A_i| = 15$. An $\times$ in the (i,j)'th entry indicates that outcome $x_i \in A_j$.

Outcomes x	$p(x)$	A_1	A_2	A_3	A_4	A_5	A_6
x_0	0.012	$\times$		$\times$		$\times$	
x_1	0.022		$\times$		$\times$		
x_2	0.023	$\times$		$\times$		$\times$	
x_3	0.033		$\times$				
x_4	0.034	$\times$				$\times$	$\times$
x_5	0.044		$\times$	$\times$		$\times$	
x_6	0.045		$\times$			$\times$	$\times$
x_7	0.055			$\times$	$\times$		$\times$
x_8	0.056	$\times$		$\times$			
x_9	0.066				$\times$	$\times$	
x_{10}	0.067		$\times$		$\times$	$\times$	
x_{11}	0.077		$\times$				
x_{12}	0.078	$\times$			$\times$		$\times$
x_{13}	0.088		$\times$				
x_{14}	0.089	$\times$		$\times$		$\times$	$\times$

Table 2.3: Description of System III with $N = 6$ and $|\cup_{i=1}^N A_i| = 15$. An $\times$ in the (i, j)'th entry indicates that outcome $x_i \in A_j$.

Outcomes x	$p(x)$	A_1	A_2	A_3	A_4	A_5	A_6	A_7
x_0	0.0329			$\times$				
x_1	0.1076	$\times$	$\times$	$\times$				$\times$
x_2	0.0599					$\times$		
x_3	0.1108			$\times$		$\times$		
x_4	0.0420		$\times$					
x_5	0.0055		$\times$	$\times$				$\times$
x_6	0.0508					$\times$	$\times$	$\times$
x_7	0.1142	$\times$				$\times$		
x_8	0.0480						$\times$	$\times$
x_9	0.0235						$\times$	$\times$
x_{10}	0.0676	$\times$	$\times$					$\times$
x_{11}	0.0295		$\times$		$\times$			
x_{12}	0.0441	$\times$		$\times$			$\times$	
x_{13}	0.1265	$\times$			$\times$		$\times$	
x_{14}	0.1058				$\times$	$\times$		$\times$

Table 2.4: Description of System IV with $N = 7$ and $|\cup_{i=1}^{N} A_i| = 15$. An $\times$ in the (i, j)'th entry indicates that outcome $x_i \in A_j$.

2.1.5 The Stepwise Lower Bound

In this section, we present an algorithmic lower bound on the probability of a finite union. The bound is a suboptimal instance of the following bound by Kounias [15], which is generally infeasible to determine exactly:

$$P(\bigcup_{i=1}^{N} A_i) \geq \max_{\mathcal{J}}\{\sum_{i\in\mathcal{J}} P(A_i) - \sum_{\substack{i,j\in\mathcal{J}\\ i<j}} P(A_i \bigcap A_j)\}, \tag{2.14}$$

where $\mathcal{J} \subseteq \mathcal{S} \triangleq \{1, 2, \ldots, N\}$. We note that the above maximization requires a search over $2^N - 1$ subsets; this results in a search complexity that increases exponentially with N. We reduce this complexity by proposing a suboptimal algorithm whose search complexity is empirically observed to be $O(N^2)$. It employs a stepwise searching technique which is commonly used in statistics to search for the optimal subset of predictors in regression problems involving a large number of candidates (see, for example [27]). In the statistical applications, the objective function is usually based on a goodness of fit measure, such as the Akaike Information Criterion (AIC) [27]. In our problem, the objective is to maximize the probability on the RHS of (2.14).

Let $M(\mathcal{J})$ denote the RHS of (2.14) for subset $\mathcal{J} \subseteq \mathcal{S}$. The basic idea of the algorithm is to (iteratively) construct a subset $\mathcal{J}$ such that both $M(\mathcal{J} \bigcup \{i\})$ and $M(\mathcal{J} \backslash \{i\})$ are not greater than $M(\mathcal{J})$ for any index $i \in \mathcal{S}$. Specifically the algorithm is as follows:

Step 0 Set $\mathcal{J}_1 = \emptyset$ and $\mathcal{J}_2 = \mathcal{S}$. Set $M(\mathcal{J}_1) = 0$ and compute $M(\mathcal{J}_2)$.

Step 1
- *1a* Augment $\mathcal{J}_1$ by adding to it (if possible) the index $i \in \mathcal{S} \backslash \mathcal{J}_1$ such that the RHS of (2.14) is maximized and $M(\mathcal{J}_1 \bigcup \{i\}) > M(\mathcal{J}_1)$.
- *1b* Shrink $\mathcal{J}_2$ by deleting from it (if possible) the index $j \in \mathcal{J}_2$ such that the RHS of (2.14) is maximized and $M(\mathcal{J}_2 \backslash \{j\}) > M(\mathcal{J}_2)$.

Step 2 Repeat Step 1 until the RHS of (2.14) can no longer be improved in both Steps 1a and 1b.

Step 3 *3a* Shrink $\mathcal{J}_1$ by deleting (if possible) the index $j \in \mathcal{J}_1$ such that the RHS of (2.14) is maximized and $M(\mathcal{J}_1 \backslash \{j\}) > M(\mathcal{J}_1)$.

3b Augment $\mathcal{J}_2$ by adding to it (if possible) the index $i \in \mathcal{S} \backslash \mathcal{J}_2$ such that the RHS of (2.14) is maximized and $M(\mathcal{J}_2 \bigcup \{i\}) > M(\mathcal{J}_2)$.

Step 4 Repeat Step 3 until the RHS of (2.14) can no longer be improved in both Steps 3a and 3b.

Step 5 Repeat Steps 1–4 until the metric M can no longer be improved. Output the stepwise lower bound as $\max\{M(\mathcal{J}_1), M(\mathcal{J}_2)\}$.

2.2 The Greedy Algorithm Upper Bound

We first define the notion of a tree based on a set of pairs of indices. Then, we state a lemma which yields our general algorithmic upper bound in terms of a tree.

Theorem 2.10 *[12]* Let $A_1, A_2, \cdots, A_N$ be N sets, where $N \geq 3$. Then

$$P\Big(\bigcup_{i=1}^{N} A_i\Big) \leq \sum_{i=1}^{N} P(A_i) - \sum_{(i,j) \in T_0} P(A_i \bigcap A_j), \tag{2.15}$$

where T_0 is any tree spanning the N indices of the sets $A_1, A_2, \cdots, A_N$ and (i, j) is an edge in T_0.

A proof of Theorem 2.10 appeared in [12, p. 31]. Next we provide another proof which is easier to understand.

Definition 1 Fix $N \geq 3$. A set of $N-1$ pairs of distinct indices

$$\{(i_1, j_1), (i_2, j_2), \ldots, (i_{N-1}, j_{N-1})\}$$

is a *tree* if there exists a permutation $(\pi(1), \pi(2), \ldots, \pi(N-1))$ of $(1, 2, \ldots, N-1)$ such that for each $k = 2, \ldots, N-1$, either

$$\text{(a)} \quad i_{\pi(k)} \in \{i_{\pi(1)}, j_{\pi(1)}, \ldots, i_{\pi(k-1)}, j_{\pi(k-1)}\}$$

or

$$\text{(b)} \quad j_{\pi(k)} \in \{i_{\pi(1)}, j_{\pi(1)}, \ldots, i_{\pi(k-1)}, j_{\pi(k-1)}\},$$

but not both (a) and (b).

The set of pairs in Definition 1 corresponds to the usual notion of a tree by setting the nodes of the tree to be the indices. Specifically, let either $i_{\pi(1)}$ or $j_{\pi(1)}$ denote the root of the tree. For $k = 2, \ldots, N-1$, attach the new node $j_{\pi(k)}$ to the previously attached node $i_{\pi(k)}$ if (a) holds, otherwise attach the new node $i_{\pi(k)}$ to the previously attached node $j_{\pi(k)}$. Note that the definition implies that there are exactly N distinct indices among the $N-1$ pairs.

Lemma 2.11 Let $A_1, \ldots, A_N$ be N sets, where $N \geq 3$. If the pairs of indices

$$\{(i_1, j_1), (i_2, j_2), \ldots, (i_{N-1}, j_{N-1})\}$$

form a tree, then

$$P\Big(\bigcup_{i=1}^{N} A_i\Big) \leq \sum_{i=1}^{N} P(A_i) - \sum_{k=1}^{N-1} P(A_{i_k} \bigcap A_{j_k}). \tag{2.16}$$

Proof: The proof uses induction. The result is easily seen to be true for $N = 3$. For general N, note that if π is a permutation of $\{1, \ldots, N-1\}$ that satisfies the conditions of Definition 1, then the set of pairs $\{(i_{\pi(1)}, j_{\pi(1)}), \ldots, (i_{\pi(N-2)}, j_{\pi(N-2)})\}$ is a tree (containing exactly $N-1$ distinct indices). Let $k^\star$ be the index ($i_{\pi(N-1)}$ or $j_{\pi(N-1)}$) not included in this smaller tree. Assume the result is true for $N-1$ sets; thus

$$P\Big(\bigcup_{\substack{i=1\\ i\neq k^\star}}^{N} A_i\Big) \leq \sum_{\substack{i=1\\ i\neq k^\star}}^{N} P(A_i) - \sum_{k=1}^{N-2} P(A_{i_{\pi(k)}} \bigcap A_{j_{\pi(k)}}). \tag{2.17}$$

Clearly, if we now include $A_{k^\star}$ into the union, we are increasing the probability by exactly $P(A_{k^\star}) - P(\bigcup_{i\neq k^\star}(A_{k^\star} \bigcap A_i))$, and since $P(\bigcup_{i\neq k^\star}(A_{k^\star} \bigcap A_i)) \geq P(A_{k^\star} \bigcap A_i)$ for any $i \neq k^\star$, we maintain an upper bound to $P(\bigcup_{i=1}^{N} A_i)$ if we add $P(A_{k^\star}) - P(A_{i_{\pi(N-1)}} \bigcap A_{j_{\pi(N-1)}})$ to the right hand side of (2.17). □

(2.16) is exactly (2.15). We next apply the Greedy Algorithm in [26, p. 17] which constructs the *optimal* spanning tree T_0 such that the second term in the right hand side of (2.15) is maximized [26][Thm. 2.2].

Greedy Algorithm: Consider a (fully) connected graph G with N vertices and $\binom{N}{2}$ edges (i,j) of weights $P(A_i \cap A_j)$. Construct a set of edges T_0 as follows.

Step 0: $T_0 = \emptyset$.

Step 1: Add to T_0 the edge with maximum weight.

Step 2: From the remaining edges, add to T_0 the edge with maximum weight subject to the constraint that T_0 remain cycle-free.

Step 3: Repeat Step 2 until T_0 contains $N-1$ edges.

We next consider the implementation complexity of this algorithm.

Given an undirected graph G with N nodes and $K=\binom{N}{2}$ edges, the complexity of the Greedy Algorithm is composed of the complexity for sorting the edges and the complexity for detecting cycles. For a fully connected graph, sorting requires $O(K\log_2 K) = O(K\log_2(N^2)) = O(K\log_2 N)$ comparisons. The time to detect a cycle depends on the method we use for this step. The naive method we employed requires $O(KN)$ comparisons [1, p. 221] resulting in $O(KN)$ complexity for the entire algorithm. In [1, pp. 521-523], an improved implementation of the Greedy algorithm for detecting cycles is given which requires $O(K + N \log_2 N)$ comparisons.

In Chapter 4, we apply this algorithm, as well as the Stepwise algorithm and the KAT lower bound, to investigate the SER and BER performance of M-PSK and M-QAM modulated signals sent over an AWGN channel, where the input bit are i.i.d. non-uniformly distributed, and the modulated signals are decoded using soft decision MAP decoding.

In the next Chapter, we determine appropriate expressions for the SER and BER of these communication systems that are given in terms of unions of events. We also obtain practical expression for the single and pairwise event probabilities.

Chapter 3

Non-uniform signaling over AWGN channels

3.1 Preamble

We consider a non-uniform independent and identically distributed (i.i.d.) binary source $\{X_i\}$, with distribution $P\{X = 0\} = p$, that is transmitted via M-PSK or M-QAM modulation with Gray mapping (cf pp. 71-72 in the Appendix) over an AWGN channel with single-sided power spectral density N_0. The justification for the non-uniformity assumption of the source is as follows. In many practical image and speech compression techniques, after some transformation, the transform coefficients are turned into bit streams (binary source). Due to the sub-optimality of the compression scheme, the bit stream often exhibits a certain amount of redundancy [3, 6, 28]. This embedded residual redundancy can be characterized by modeling the bit stream as an i.i.d. non-uniform process or as a Markov process [2, 3, 6]. The source stream is grouped in blocks of $\log_2 M$ bits which are each subsequently mapped to a two dimentional modulation signal for transmission over the channel. At the receiver, maximum a posteriori decoding (MAP) is performed in estimating the transmitted M-ary signal since MAP decoding is optimal in the sense that it minimizes the symbol error rate. More specifically, if one of M signals $s_1, s_2, \ldots, s_M$ is sent, then the MAP decoder declares that s_k was sent if, for $i = 1, 2, \ldots, M$ and $i \neq k$, the MAP metric of s_k is bigger than the metric of s_i; i.e.,

$$P\left(s_k|r\right) \geq P\left(s_i|r\right),$$

where $r = s_u + n$ is the received signal , s_u is the actual signal sent, and n is Gaussian distributed noise with 0-mean and covariance matrix $\frac{N_0}{2} I_2$, where I_2 is the 2×2 identity matrix.

3.2 Symbol Error Rate

The probability of symbol decoding error P_s is

$$P_s = \sum_{u=1}^{M} P(\epsilon|s_u)P(s_u) = \sum_{u=1}^{M} P\Big(\bigcup_{i\neq u} \epsilon_{iu}\Big)P(s_u), \tag{3.1}$$

where $P(\epsilon|s_u)$ is the conditional probability of error given that s_u was sent, and ϵ_{iu} represents the event that s_i has a higher MAP metric than s_u given that s_u was sent. Therefore,

$$P\left(\epsilon_{iu}\right) = Pr\{p(r|s_i)p(s_i) \geq p(r|s_u)p(s_u)\} \tag{3.2}$$

$$\begin{aligned} P\left(\epsilon_{iu} \cap \epsilon_{ju}\right) = Pr\{&p(r|s_i)p(s_i) \geq p(r|s_u)p(s_u), \\ &p(r|s_j)p(s_j) \geq p(r|s_u)p(s_u)\} \end{aligned}$$

Since n is Gaussian distributed with mean 0 and covariance $\frac{N_0}{2} I_2$, i.e.,

$$p(n) = \frac{1}{\pi N_0} e^{-||n||^2/N_0},$$

then

$$p(r|s_i) = \frac{1}{\pi N_0} e^{-||r-s_i||^2/N_0}.$$

The right hand side of equation (3.2) can be easily reduced to:

$$Pr\{N_0 \ln p(s_i) - ||n + s_u - s_i||^2 \geq N_0 \ln p(s_u) - ||n||^2\}$$

which is

$$Pr\left\{\frac{\sqrt{2}<n, s_i - s_u>}{\sqrt{N_0}||s_i - s_u||} \geq \frac{||s_i - s_u||}{\sqrt{2N_0}} + \frac{N_0 \ln \frac{p(s_u)}{p(s_i)}}{\sqrt{2N_0}||s_i - s_u||}\right\},$$

where $||\cdot||$ is the Euclidean norm and $<\cdot,\cdot>$ denotes the usual dot product. Notice that

$$X_i = \frac{\sqrt{2}}{\sqrt{N_0}||s_i - s_u||}<n, s_i - s_u>$$

is Gaussian with 0-mean, unit variance. Then

$$P\left(\epsilon_{iu}\right) = Q\left(\frac{d_{iu}}{\sqrt{2N_0}} + \frac{\sqrt{2N_0} \ln \frac{p(s_u)}{p(s_i)}}{2d_{iu}}\right),$$

where

$$d_{iu} = ||s_i - s_u||,$$

$$Q\left(x\right) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-y^2/2\right) dy$$

and similarly

$$\begin{aligned} P\left(\epsilon_{iu} \cap \epsilon_{ju}\right) = \Psi\Bigg(\rho_{iju}, &\frac{d_{iu}}{\sqrt{2N_0}} + \frac{\sqrt{2N_0} \ln \frac{p(s_u)}{p(s_i)}}{2d_{iu}}, \\ &\frac{d_{ju}}{\sqrt{2N_0}} + \frac{\sqrt{2N_0} \ln \frac{p(s_u)}{p(s_i)}}{2d_{ju}}\Bigg), \end{aligned} \tag{3.3}$$

where

$$\rho_{iju} = E[X_i X_j] = \frac{<s_i - s_u, s_j - s_u>}{||s_i - s_u|| \cdot ||s_j - s_u||}$$

is the covariance of X_i and X_j,

$$\Psi(\rho, a, b) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_a^\infty \int_b^\infty e^{-\frac{\left(x^2 - 2\rho x y + y^2\right)}{2(1-\rho^2)}} dx\ dy.$$

If we apply Theorem 2.2, inequality (2.14) with the Stepwise searching technique and Theorem 2.10 (along with the Greedy Algorithm) to $P(\bigcup_{i\neq u} \epsilon_{iu})$ and substitute in (3.1), we obtain two lower bounds and one upper bound on P_s in terms of $P(\epsilon_{iu})$, $P(\epsilon_{iu} \cap \epsilon_{ju})$ and $P(s_u)$.

3.3 Bit Error Rate

In many cases, BER is a more useful performance measure. Under the symbol MAP decoding criterion, the bit error probability P_b can be written as

$$P_b = \sum_{u=1}^{M} P_b(u) P(s_u),$$

where

$$\begin{aligned} P_b(u) &= \frac{1}{\log_2 M} E(\texttt{\# of bit errors} | s_u \texttt{ is sent}) \\ &= \frac{1}{\log_2 M} \sum_{j=1}^{M} d(c_j, c_u) A_{j/u}, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} A_{j/u} = &\ P(s_j \texttt{ is decoded} | s_u \texttt{ is sent}) \\ = &\ 1 - P\Big(\bigcup_{i=1, i\neq j}^{M} \{P(s_i|r) \geq P(s_j|r) | s_u \texttt{ is sent}\}\Big) \\ = &\ 1 - P\Big(\bigcup_{i=1, i\neq j}^{M} \epsilon_{iju}\Big), \end{aligned} \tag{3.5}$$

where $u = 1, \ldots, M$, c_j and c_u are the (Gray mapping) bit assignments for signals s_j and s_u, respectively, $d(c_j, c_u)$ is the Hamming distance between c_j and c_u, and ϵ_{iju}

represents the event that symbol s_i has a higher metric than symbol s_j given that symbol s_u was sent. Similar to the case for the symbol error probability,

$$\begin{aligned}
P(\epsilon_{iju}) &= Pr\{p(r|s_i)p(s_i) \geq p(r|s_j)p(s_j)|s_u \texttt{ is sent}\} \\
&= Pr\{N_0 \ln p(s_i) - ||n + s_u - s_i||^2 \geq N_0 \ln p(s_j) - ||n + s_u - s_j||^2\} \\
&= Pr\{2{<}n, s_u - s_j{>} - 2{<}n, s_u - s_i{>} \\
&\qquad \geq ||s_i - s_u||^2 - ||s_j - s_u||^2 + N_0 \ln \frac{p(s_j)}{p(s_i)}\} \\
&= Pr\left\{\frac{\sqrt{2}{<}n, s_i - s_j{>}}{\sqrt{N_0}||s_i - s_j||} \geq \frac{N_0 \ln \frac{p(s_j)}{p(s_i)}}{\sqrt{2N_0}||s_i - s_j||} + \frac{||s_i - s_u||^2}{\sqrt{2N_0}||s_i - s_j||}\right. \\
&\qquad \left. - \frac{||s_j - s_u||^2}{\sqrt{2N_0}||s_i - s_j||}\right\} \\
&= Q\left(\frac{N_0 \ln \frac{p(s_j)}{p(s_i)}}{\sqrt{2N_0}||s_i - s_j||} + \frac{||s_i - s_u||^2}{\sqrt{2N_0}||s_i - s_j||} - \frac{||s_j - s_u||^2}{\sqrt{2N_0}||s_i - s_j||}\right) \qquad (3.6)
\end{aligned}$$

and

$$\begin{aligned}
P(\epsilon_{iju} \cap \epsilon_{kju}) &= Pr\{p(r|s_i)p(s_i) \geq p(r|s_j)p(s_j); p(r|\mathbf{s_k})p(\mathbf{s_k}) \geq p(r|s_j)p(s_j)\} \\
&= \Psi\left(\rho_{ikj}, \frac{N_0 \ln \frac{p(s_j)}{p(s_i)}}{\sqrt{2N_0}||s_i - s_j||} + \frac{||s_i - s_u||^2}{\sqrt{2N_0}||s_i - s_j||} - \frac{||s_j - s_u||^2}{\sqrt{2N_0}||s_i - s_j||},\right. \\
&\qquad \left. \frac{N_0 \ln \frac{p(s_j)}{p(s_k)}}{\sqrt{2N_0}||s_k - s_j||} + \frac{||s_k - s_u||^2}{\sqrt{2N_0}||s_k - s_j||} - \frac{||s_j - s_u||^2}{\sqrt{2N_0}||s_k - s_j||}\right), \quad (3.7)
\end{aligned}$$

where

$$\rho_{ikj} = E[X_i X_k] = \frac{{<}s_i - s_j, s_k - s_j{>}}{||s_i - s_j|| \cdot ||s_k - s_j||},$$

$$X_i = \frac{\sqrt{2}<n, s_i - s_j>}{\sqrt{N_0}||s_i - s_j||}$$

and

$$X_k = \frac{\sqrt{2}<n, s_k - s_j>}{\sqrt{N_0}||s_k - s_j||}.$$

Finally, applying Theorem 2.2, inequality (2.14) (with Stepwise Search Algorithm) and Theorem 2.10 (with the Greedy Algorithm) to $P(\bigcup_{i \neq j} \epsilon_{iju})$ in (3.5) yields two upper bounds and a lower bound to the bit error probability P_b.

Chapter 4

Numerical Results and Discussion

In this chapter, we investigate the applications of the results of Chapter 2 and 3 to uniform as well as non-uniform signals with 8-PSK, 16-PSK, 32-PSK and 16-QAM modulation over AWGN channels. The bounds for SER and BER are investigated when p=0.5, 0.7 and 0.9, where p is the probability that zero is input in the source. The SER and BER are plotted in terms of the SNR, E_b/N_0, where E_b is the energy per information bit and the SNR ranges from -4 dB to w dB, with w ranging from 4 dB to 14 dB depending on the modulation technique we are dealing with. That range is actually for very noisy channels where the usual union bound performs poorly. We also provide the simulation results or exact values for all the cases. For the symbol error rate, we provide the union upper bound, the Greedy algorithm upper bound, the Stepwise lower bound and the KAT lower bound. However, we do not show the union bound and the KAT bound in BER plots since the union bound is weaker and the KAT bound performs similarly as it does in SER in comparison with the Stepwise bound.

We apply the MAP decoding criterion for all the cases, i.e., for both P_s and P_b when p=0.5, 0.7 and 0.9. We note that for p=0.5, the MAP decision criterion reduces to ML criterion and there exist exact formulas to compute P_s and P_b for M-PSK and 16-QAM [21, 14]. Thus bounds are unnecessary. However, on the other hand, from the figures of P_s and P_b for M-PSK (M=8, 16, 32) and 16-QAM, we note that the Stepwise bound and the Greedy Algorithm bound agree with the exact values perfectly over a wide E_b/N_0 range (noisy channel), so these bounds could be viewed as alternative ways to compute P_s and P_b when the exact values, or formulas are computationally intensive or not available. In the figures for $p = 0.7$ and $p = 0.9$, the curves of the Stepwise bound and the Greedy Algorithm bound usually agree very closely with the simulation curves. Moreover, these curves illustrate the fact that the system performance is considerably improved by using MAP decoding instead of ML decoding.

As mentioned, exact methods are available in the case of uniform signals. The use of these bounds provide equally precise probabilities in a practical sense. Their added value is that the same code can be used to estimate performance for non-uniform signals. For the case of non-uniform signaling, we have found no published results for calculating P_s or P_b exactly; thus bounds are very useful. In Figures 4.1 to 4.24, the bounds for the symbol and bit error rates are plotted for $p = 0.5, 0.7$ and 0.9. In all cases, the stepwise and greedy bounds provide excellent results, being very close to exact values over the entire range of SNR values. The union upper bound for SER, as expected, does not perform well at lower SNR, particularly for the 32-PSK case in Figure 4.12.

The KAT lower bound for the SER provides very good results, though not as good as the stepwise lower bound for SER, and the accuracy seems to degrade as M increases in the case of M-PSK modulation. It should be remembered, however, that the KAT lower bound is a single formula which, given the single and pairwise event error probabilities, does not involve a search over a large class of bounds, and so may be more practical for applications which involve a very large number of events in the union of interest; for example, in analyzing code or bit error rates in complex coded systems (cf [23]). For the applications considered in this thesis, all bounds were practical to compute. For the SER curves, for a given value of SNR, and including the time to calculate all the single and pairwise event probabilities, the combined computing time to calculate all bounds (KAT, stepwise, greedy, and union) was less than a second for 8-PSK and 1 or 2 seconds for 32-PSK, on a computer with a 266 MHz MMX Pentium chip running Linux. For the P_b plots (described below), the corresponding times for just the KAT upper bound (computed but not shown in this paper) were less than a second even for 32-PSK, while the times for the stepwise or greedy bounds were about 1 second for 8-PSK and up to 2 minutes for 32-PSK.

We next mention the algorithms used to compute the Q and ψ functions in Chapter 3. For $Q(\cdot)$, we adopted the algorithm developed by N. Beaulieu [5], which takes only a few lines of code and which we have found to be very efficient and accurate. For the $\Psi(\cdot)$ function, we adopted an algorithm for computing Bivariate Normal tail probabilities written in Fortran by Donnelly [11] (see also [22]). We note that naive computation of the double integral in the $\Psi(\cdot)$ function by simple numerical integration rules results in a very inefficient algorithm that takes over a day of computing to produce, for example, the values for the SER curves for 16-PSK in Figure 4.6. For nonnegative lower limits, Simon and Divsalar [24] (see also [25]) provide an alternate representation of the Ψ function as the sum of two single integrals over finite ranges. For uniform signaling, the lower limits in (3.3) or (3.7) are both nonnegative, but for non-uniform signaling, one or both of these limits can be negative, in which case the Ψ function can be expressed as a linear combination of four instances of the Ψ function with nonnegative lower limits. However, with the application of Simpson's composite numerical integration rule, this method proved much less efficient for a given accuracy as compared with Donnelly's algorithm, which is based on rational function approximations. Thus the latter method was adopted.

In the course of programming, the method we use to calculate the average symbol energy E_s and the average bit energy E_b for M-QAM is worth mentioning. Given that E_{si} is the energy for signal s_i (the square of the Euclidean distance from the point s_i in the constellation to the origin), $i = 1, 2, \ldots, M$, then E_s and E_b are obtained as follows:

$$E_s = \sum_{i=1}^{M} E_{si} P(s_i)$$

and

$$E_b = \frac{E_s}{\log_2 M}$$

where $P(s_i)$ is the probability associated with signal s_i. Note that when all the signals are uniform, then E_s reduces to the following familiar expression [21] : $E_s = \frac{1}{M}\sum_{i=1}^{M} E_{si} = \frac{M-1}{6}d^2$, where d is the distance between adjacent signals.

As for M-PSK, since all the signals are on the same circle (for example, 8-PSK (A.1)), E_{si} is the same for all the signals. Using a similar formula as for M-QAM, we obtain the following expressions for E_s and E_b for M-PSK signals:

$$E_s = \sum_{i=1}^{M} E_{si} P(s_i) = E_{si}$$

and

$$E_b = \frac{E_s}{\log_2 M} = \frac{E_{si}}{\log_2 M},$$

where $i = 1, \ldots, M$.

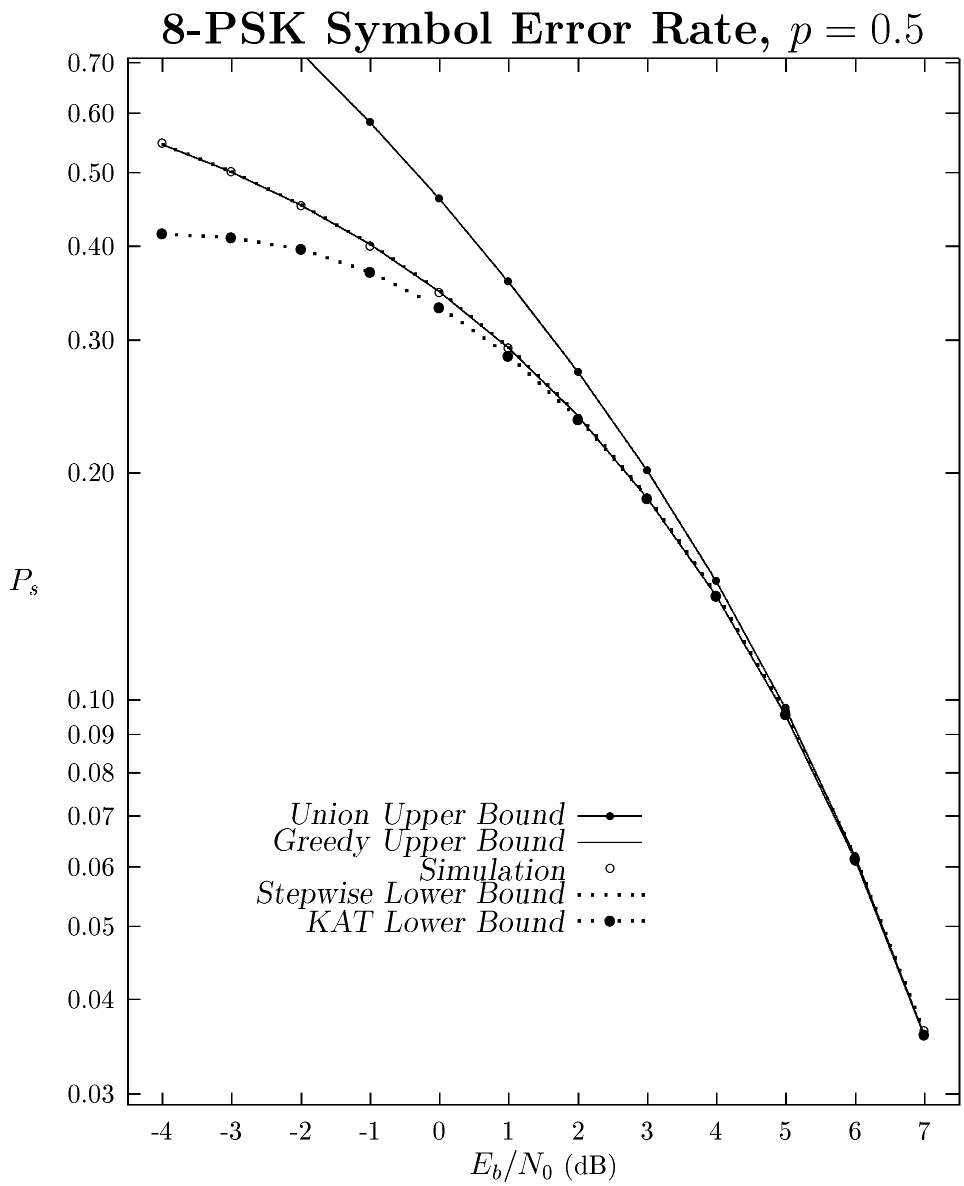

Figure 4.1: Symbol Error Rate P_s for 8-PSK, $p = 0.5$.

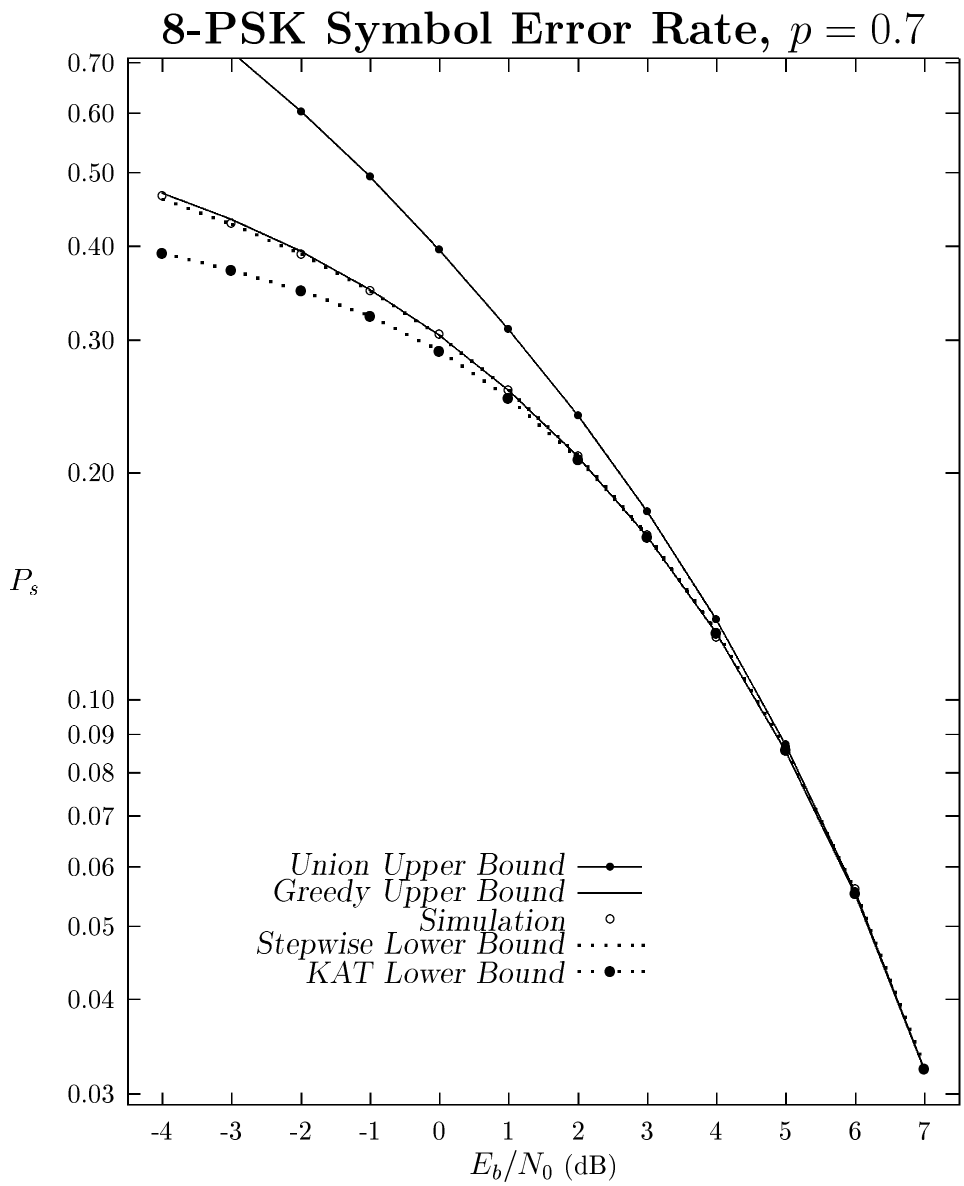

Figure 4.2: Symbol Error Rate P_s for 8-PSK, $p = 0.7$.

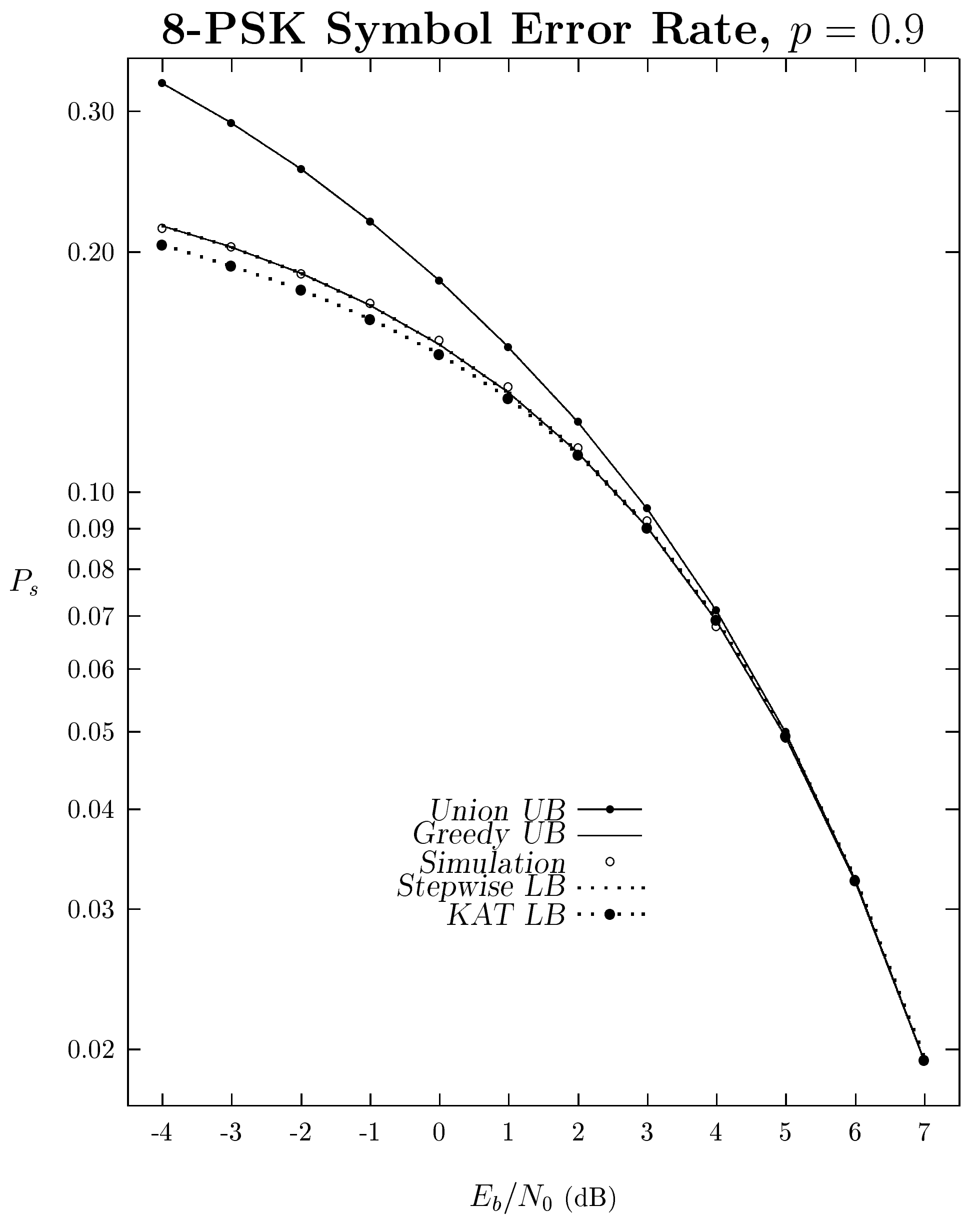

Figure 4.3: Symbol Error Rate P_s for 8-PSK, $p = 0.9$.

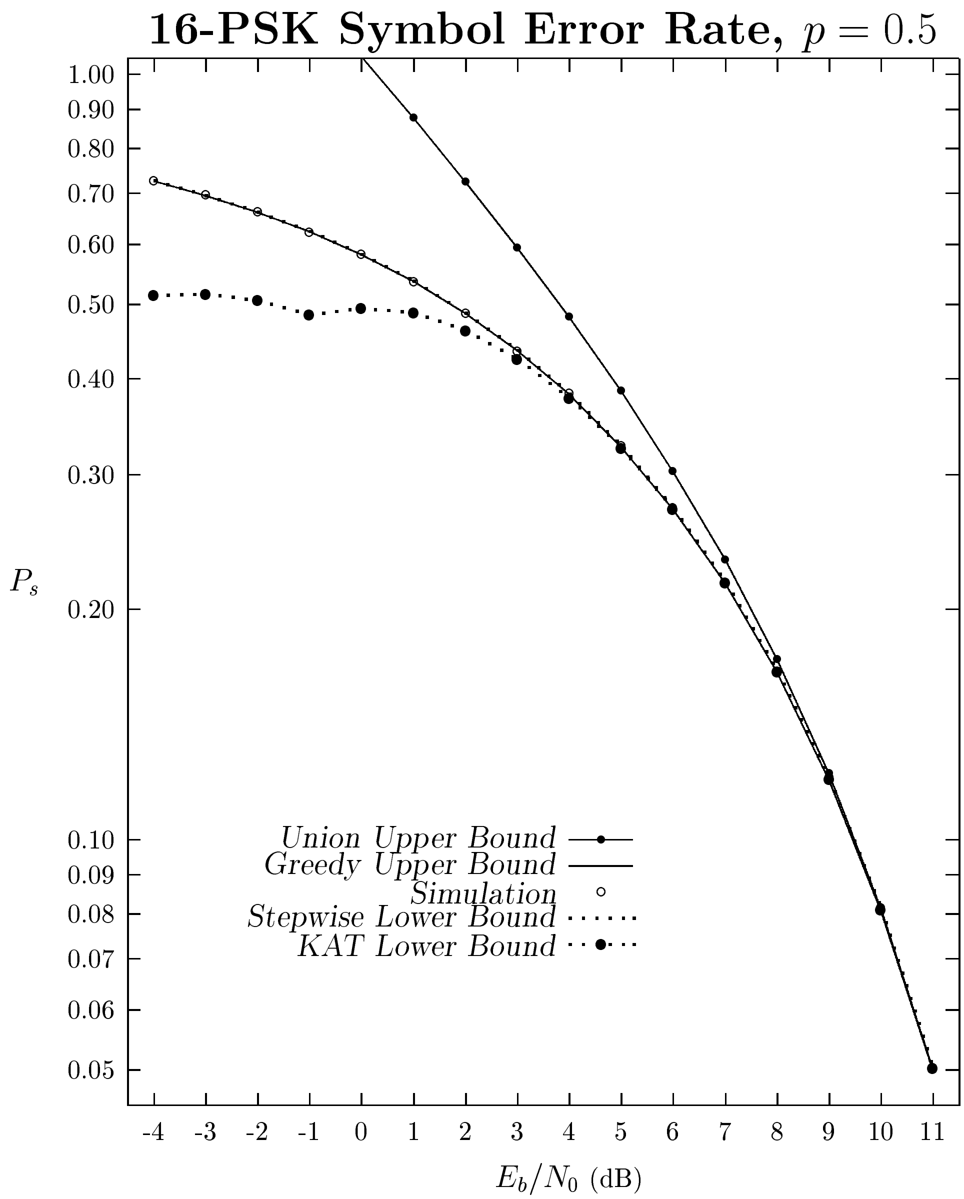

Figure 4.4: Symbol Error Rate P_s for 16-PSK, $p = 0.5$.

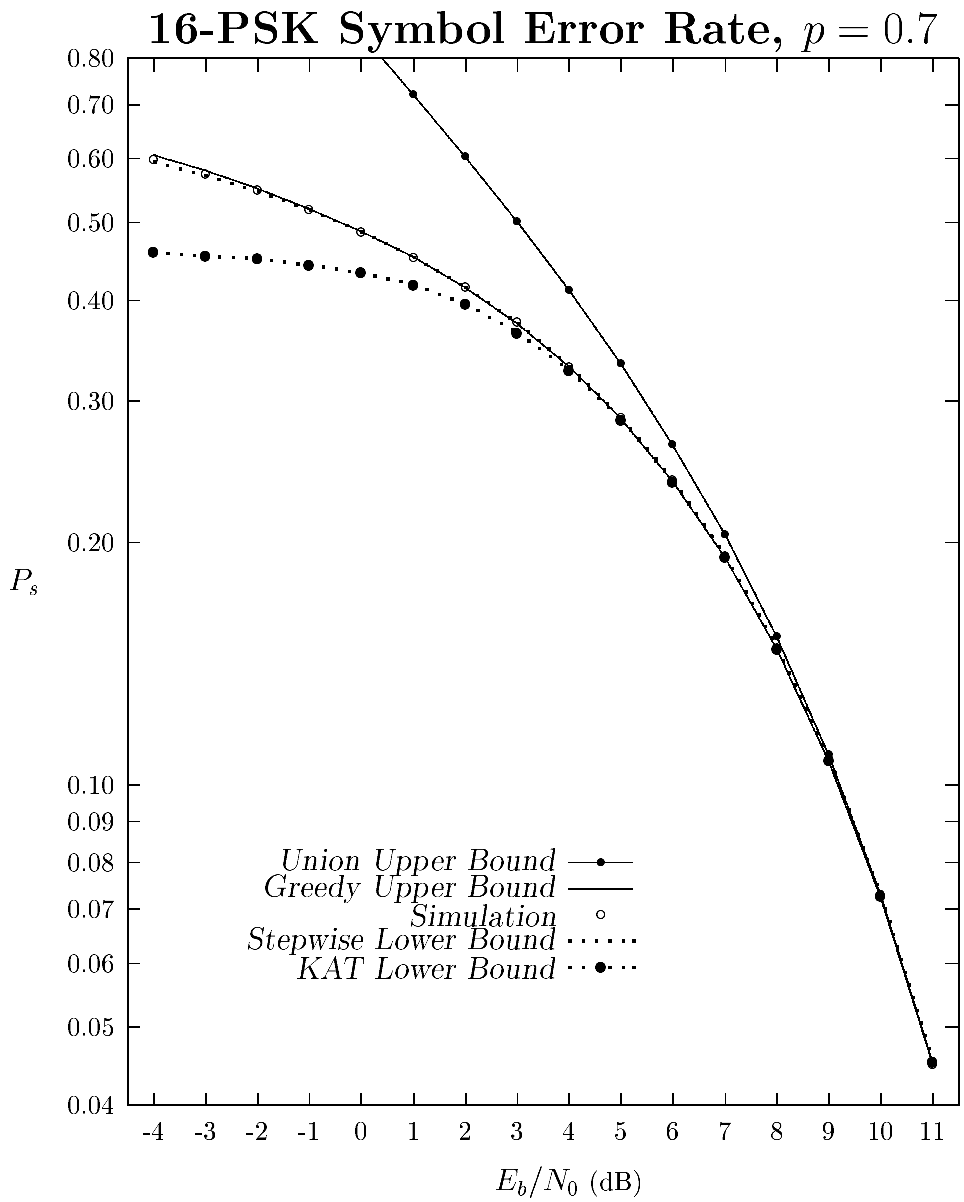

Figure 4.5: Symbol Error Rate P_s for 16-PSK, $p = 0.7$.

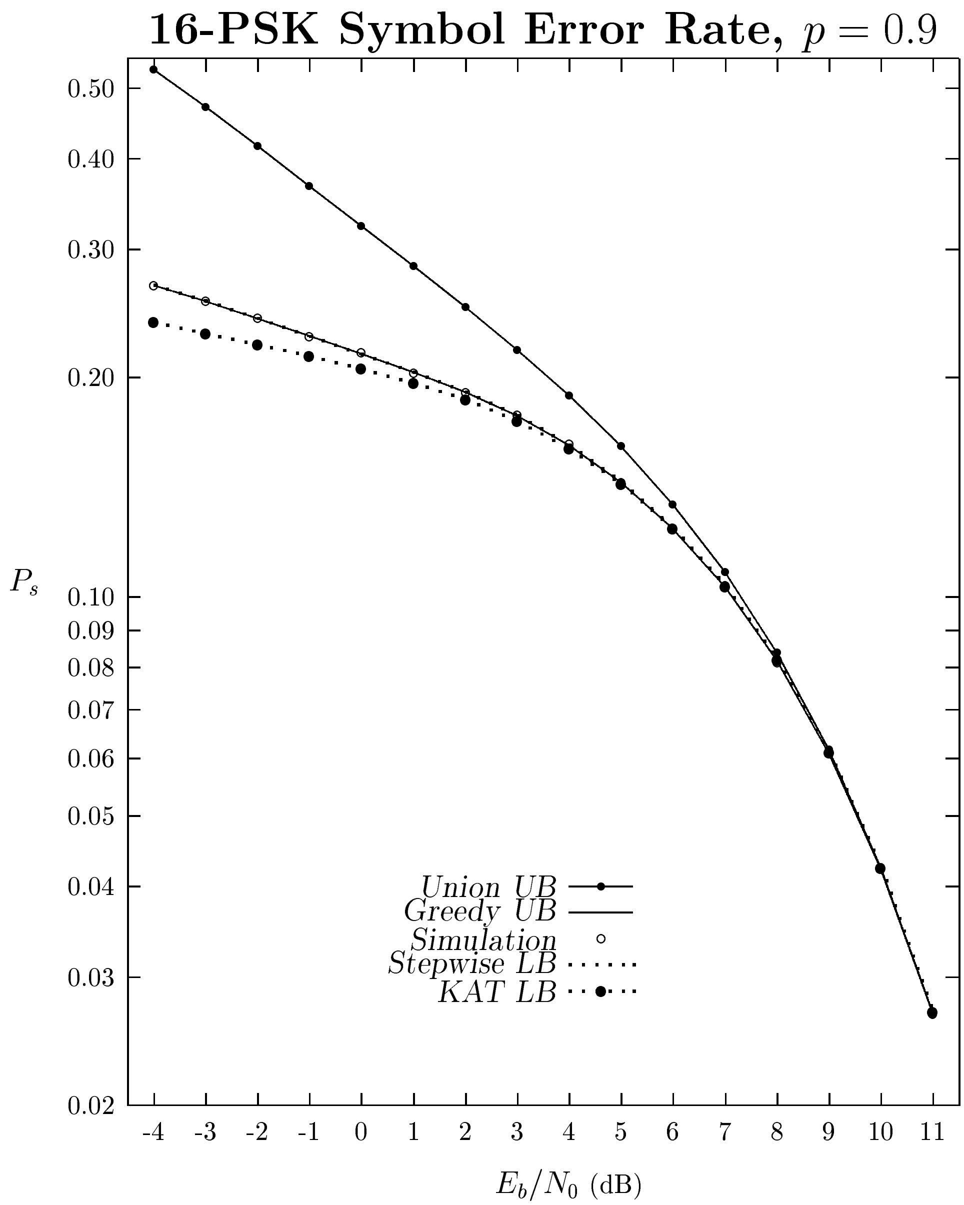

Figure 4.6: Symbol Error Rate P_s for 16-PSK, $p = 0.9$.

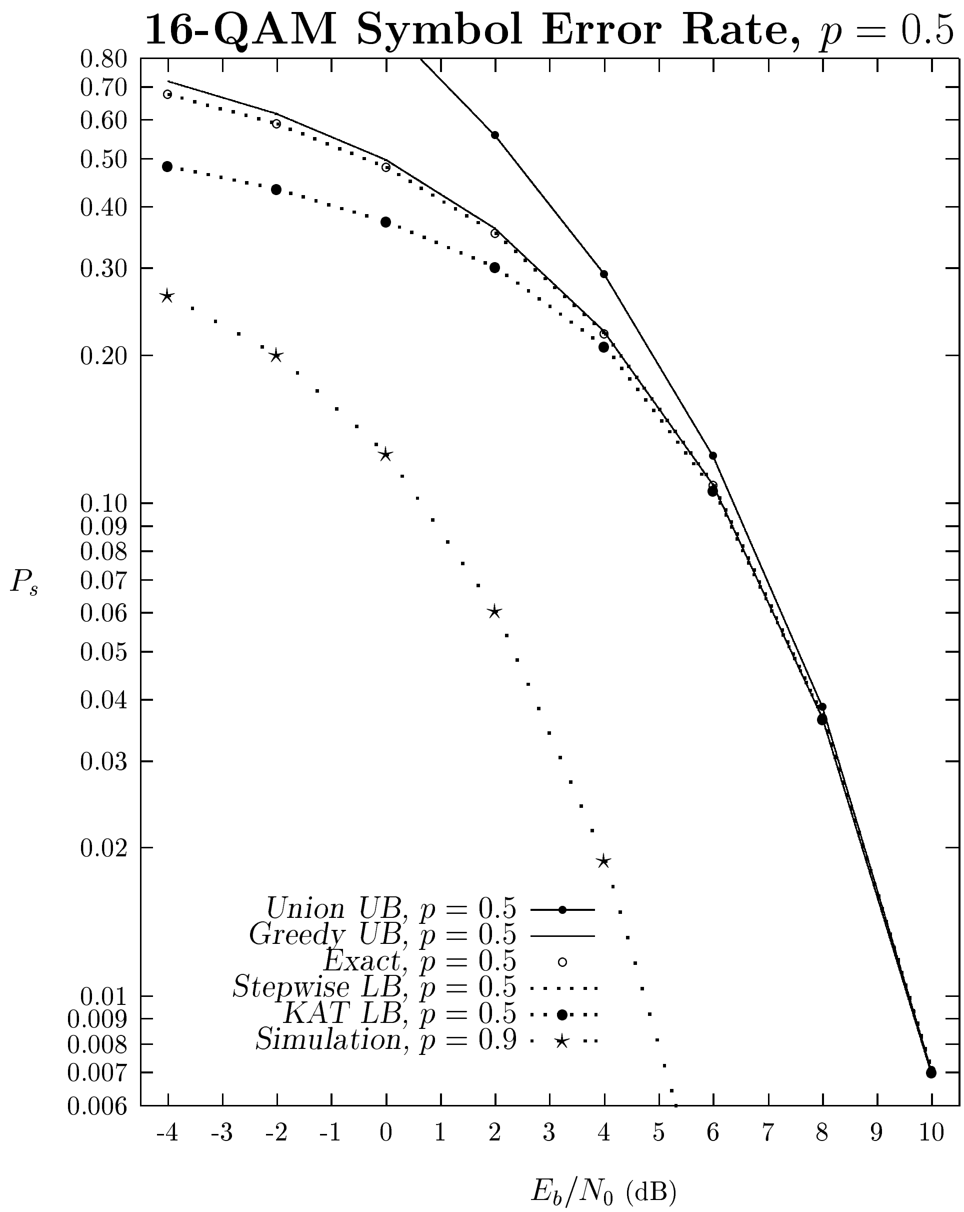

Figure 4.7: Symbol Error Rate P_s for 16-QAM, $p = 0.5$

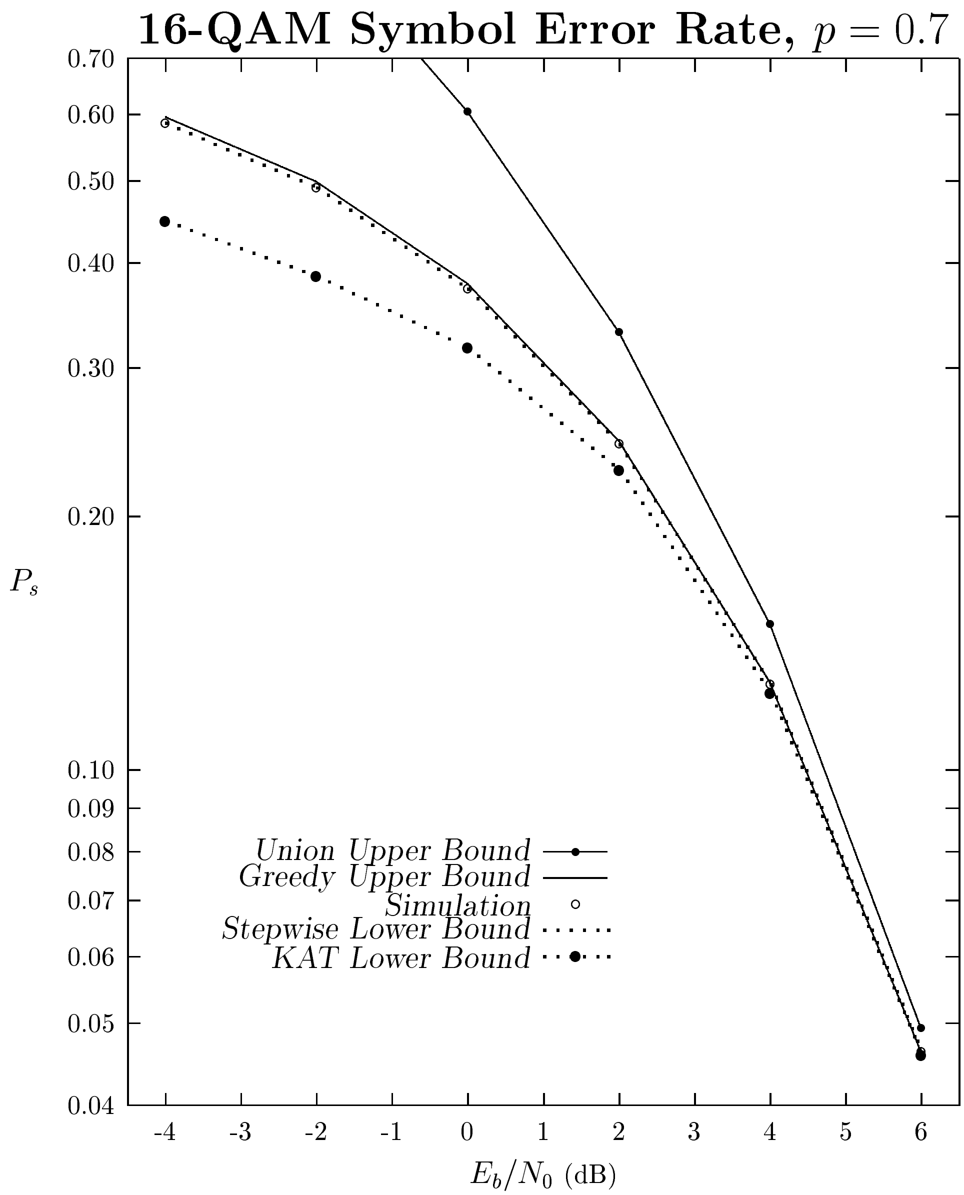

Figure 4.8: Symbol Error Rate P_s for 16-QAM, $p = 0.7$

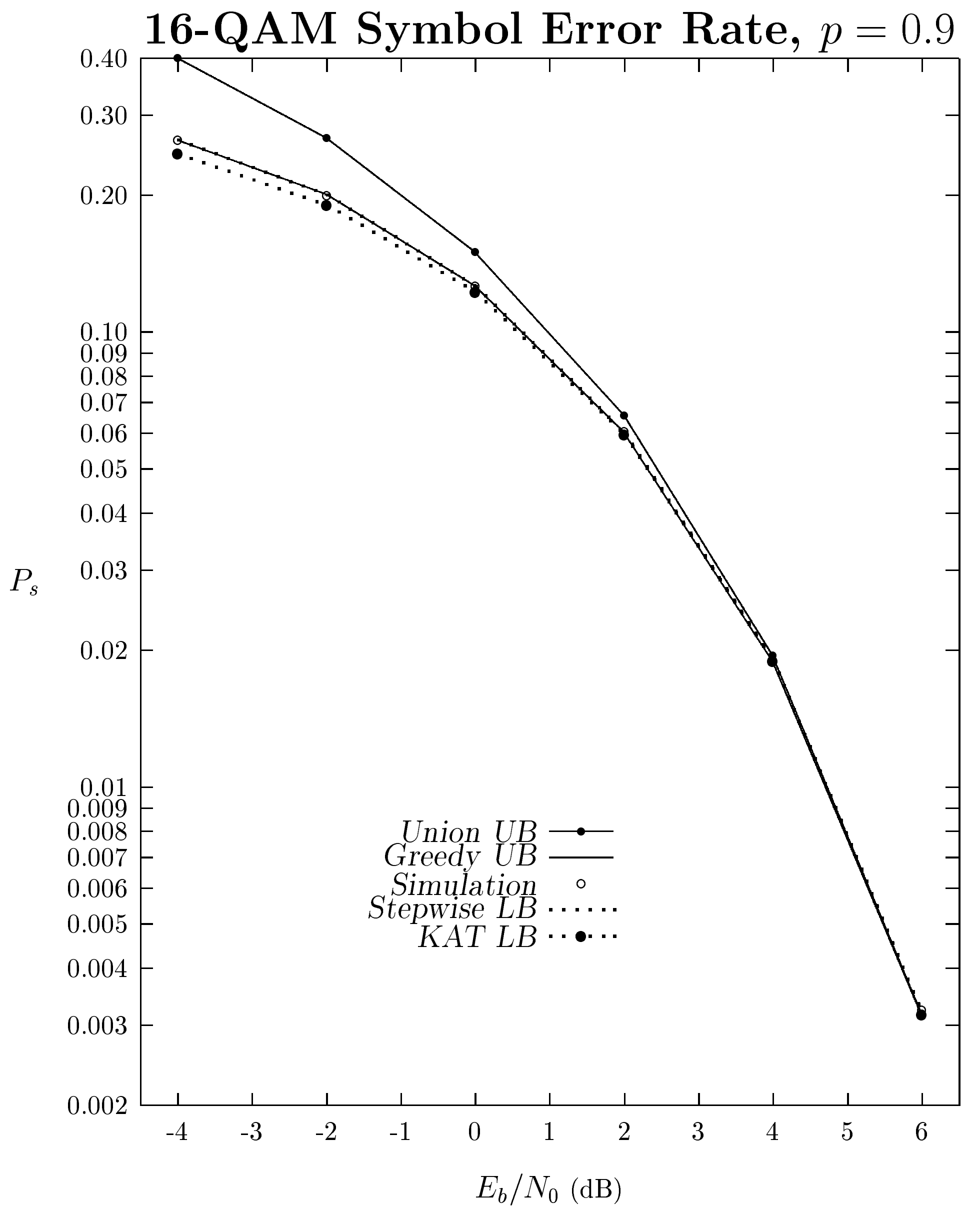

Figure 4.9: Symbol Error Rate P_s for 16-QAM, $p = 0.9$

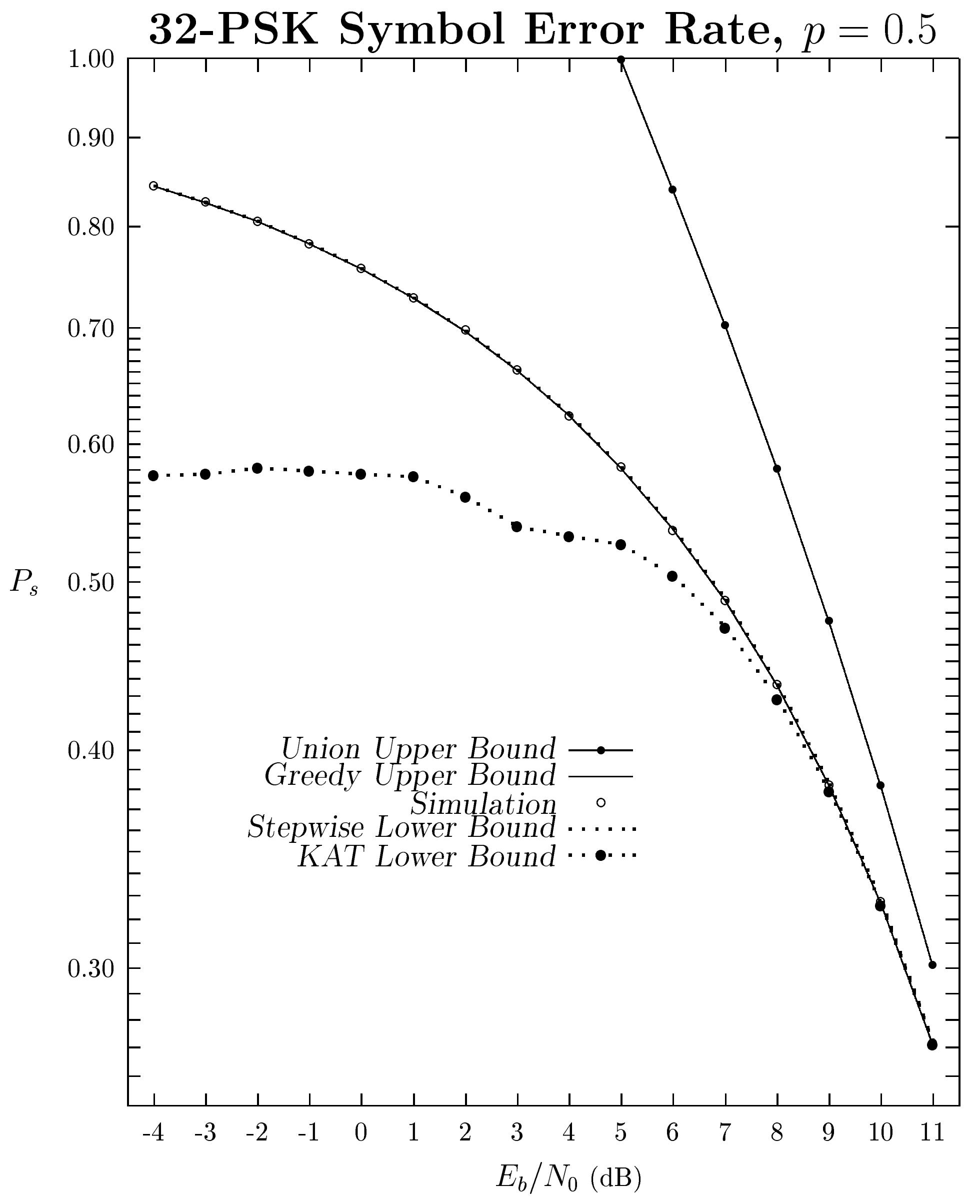

Figure 4.10: Symbol Error Rate P_s for 32-PSK, $p = 0.5$.

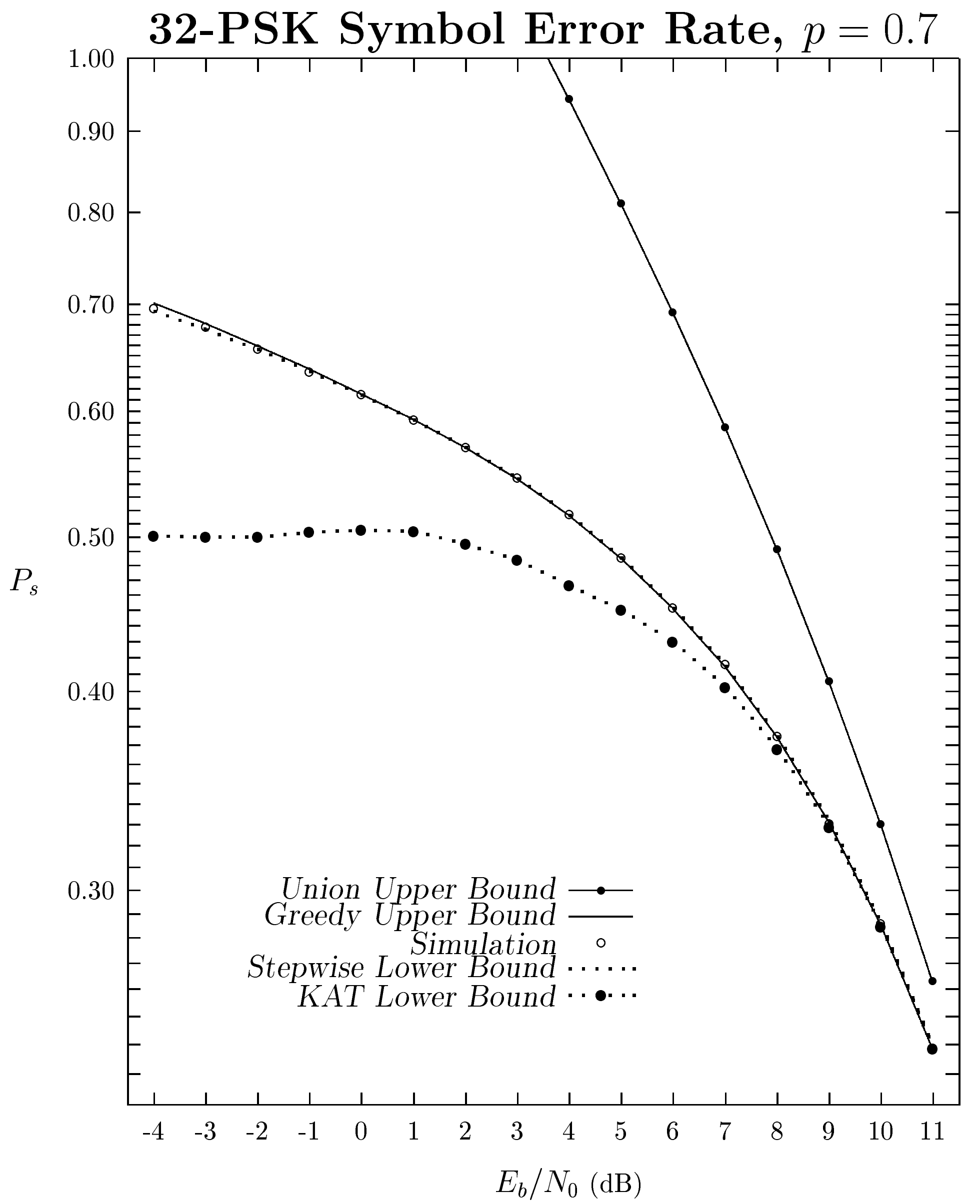

Figure 4.11: Symbol Error Rate P_s for 32-PSK, $p = 0.7$.

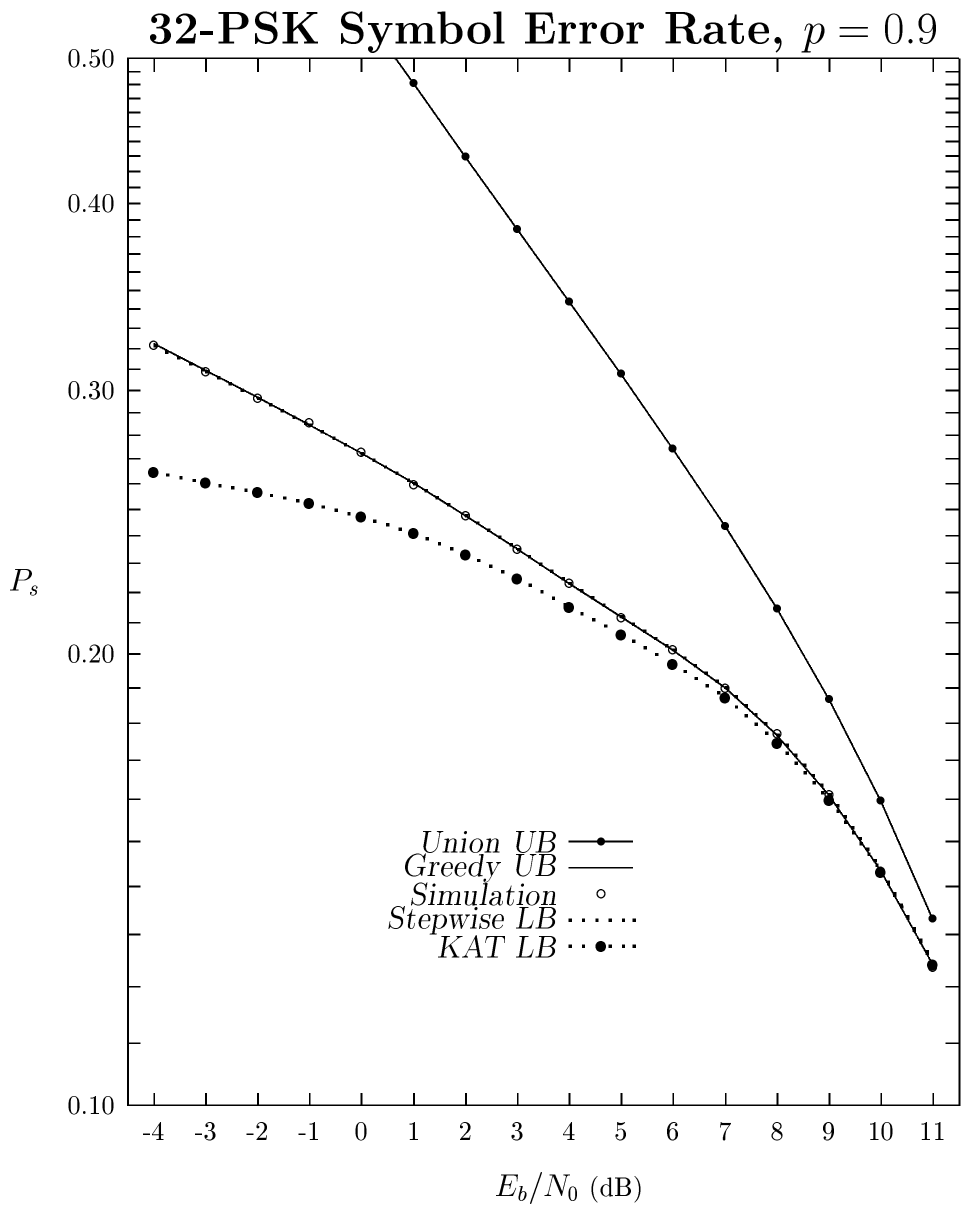

Figure 4.12: Symbol Error Rate P_s for 32-PSK, $p = 0.9$.

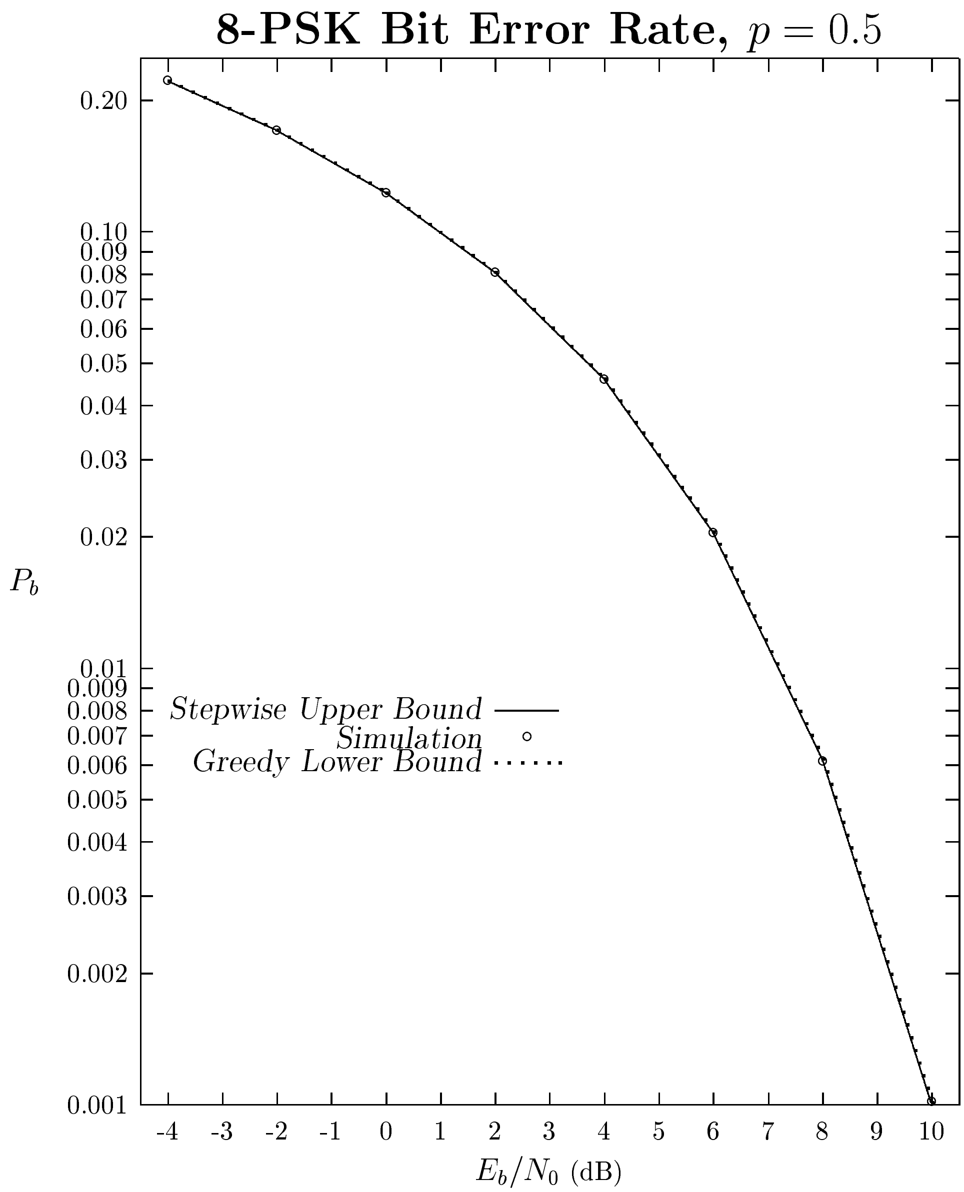

Figure 4.13: Bit Error Rate P_b for 8-PSK, $p = 0.5$.

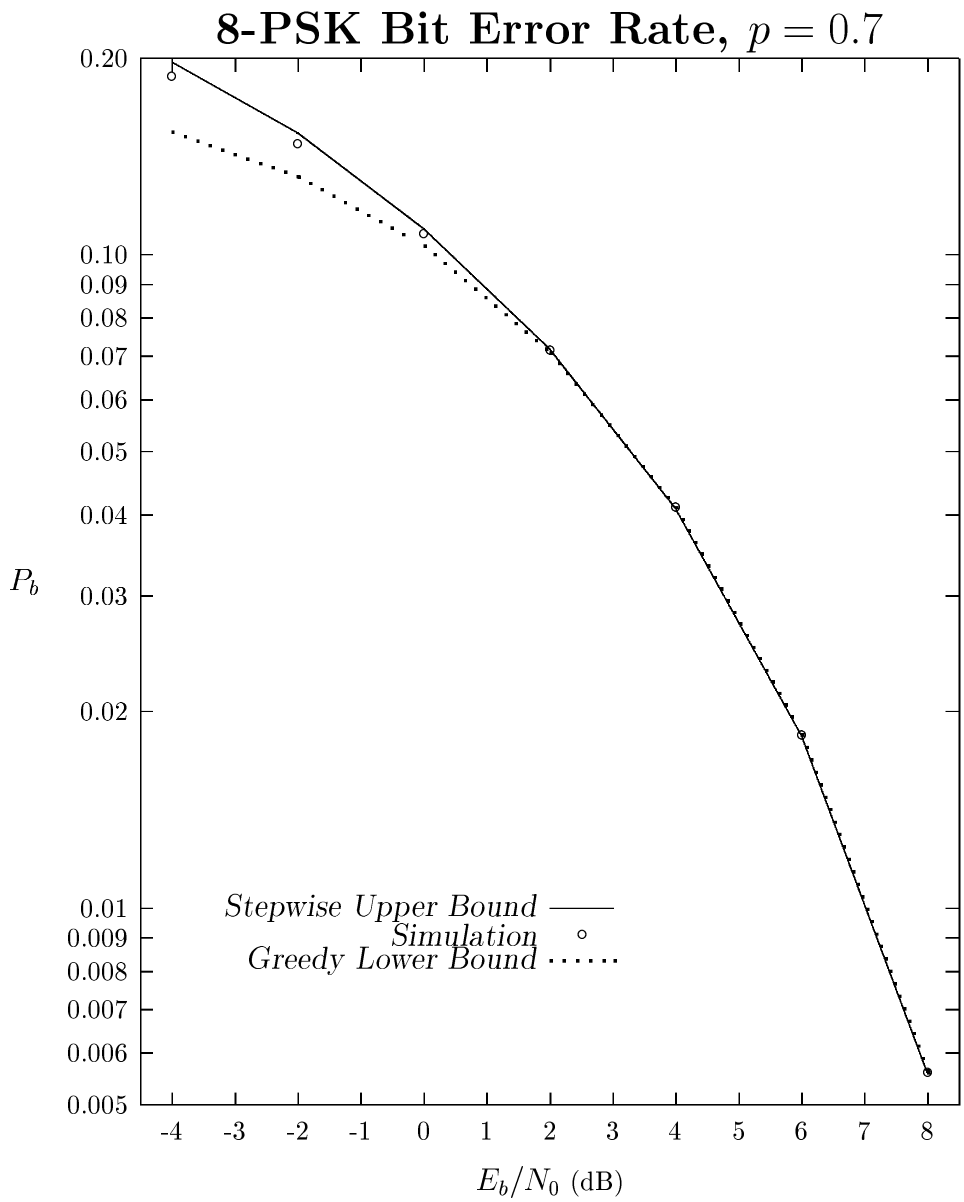

Figure 4.14: Bit Error Rate P_b for 8-PSK, $p = 0.7$.

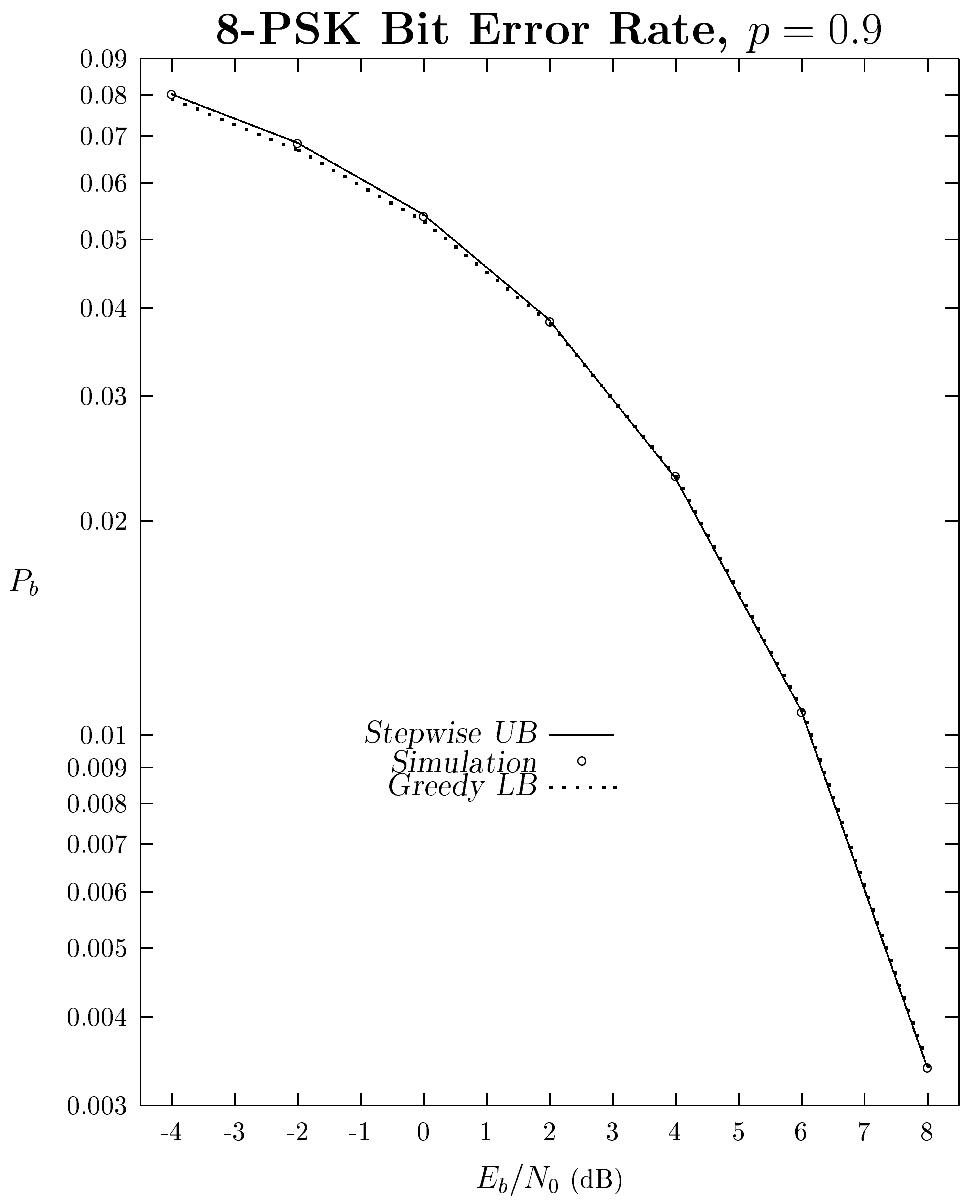

Figure 4.15: Bit Error Rate P_b for 8-PSK, $p = 0.9$.

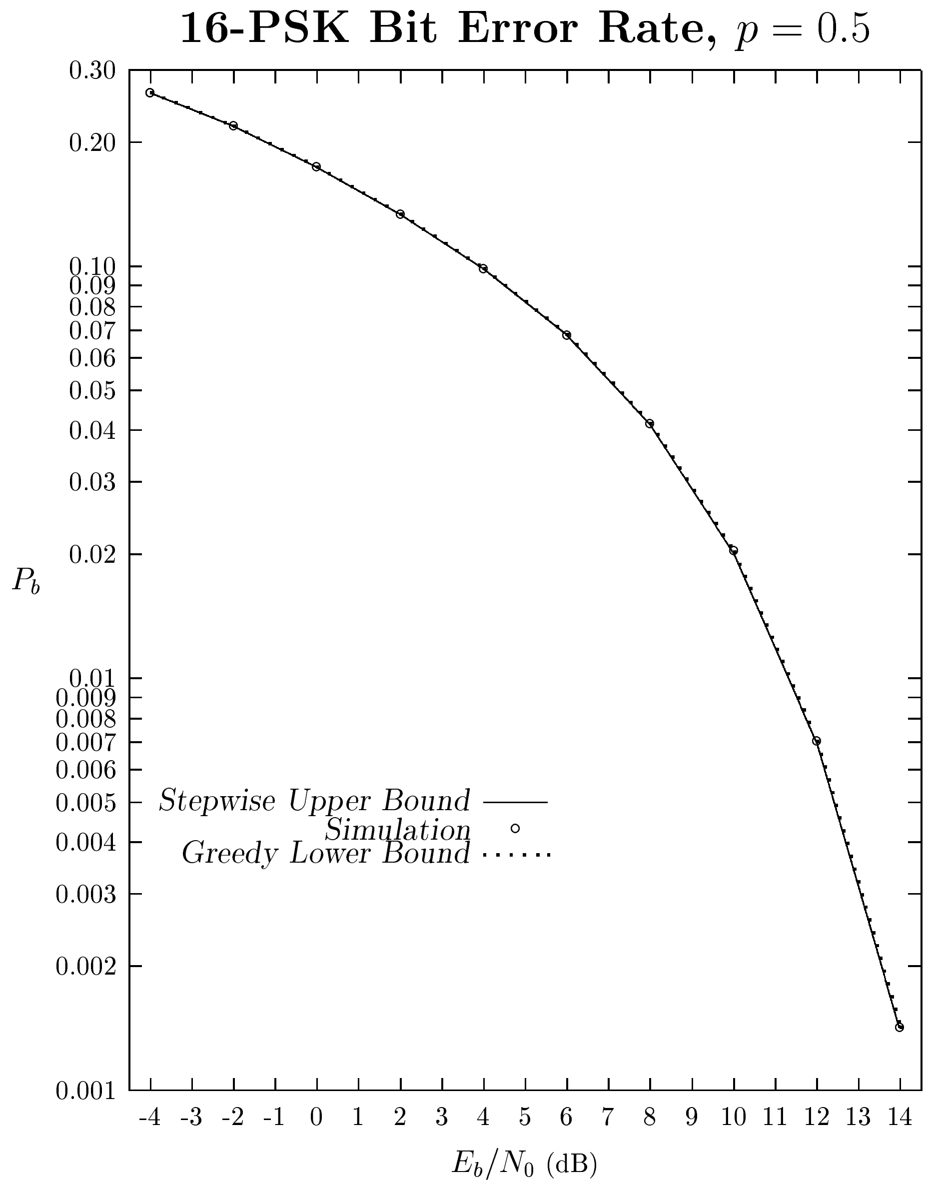

Figure 4.16: Bit Error Rate P_b for 16-PSK, $p = 0.5$.

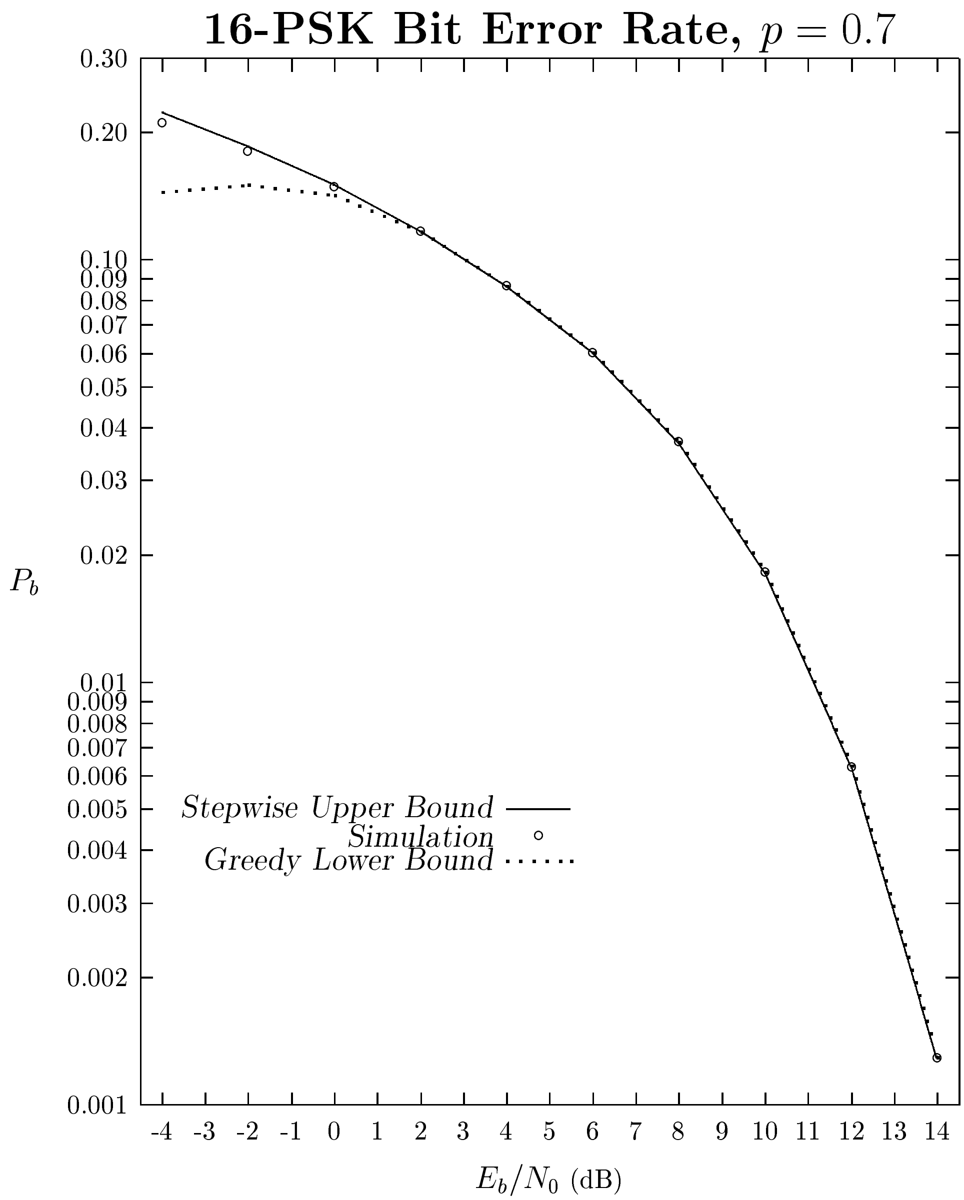

Figure 4.17: Bit Error Rate P_b for 16-PSK, $p = 0.7$.

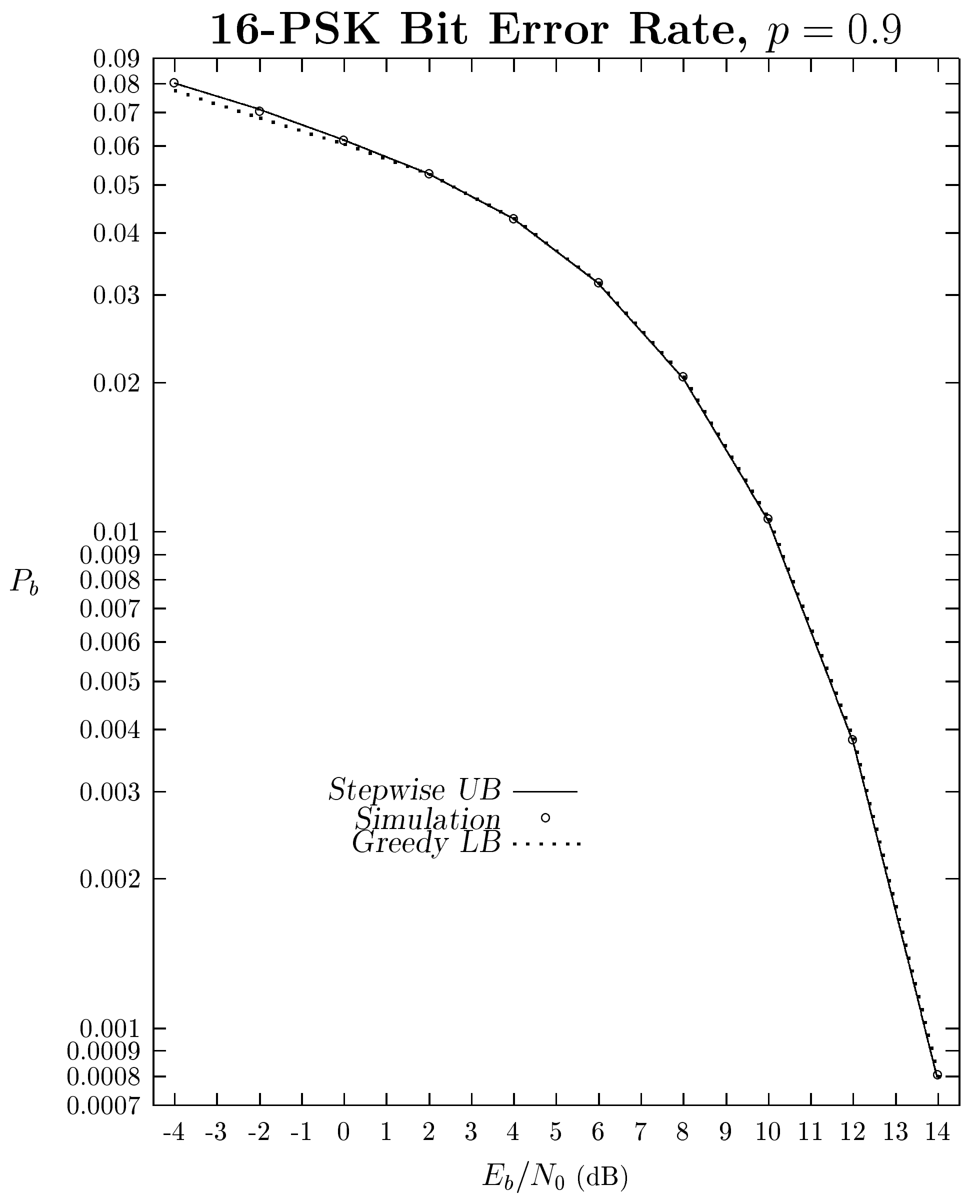

Figure 4.18: Bit Error Rate P_b for 16-PSK, $p = 0.9$.

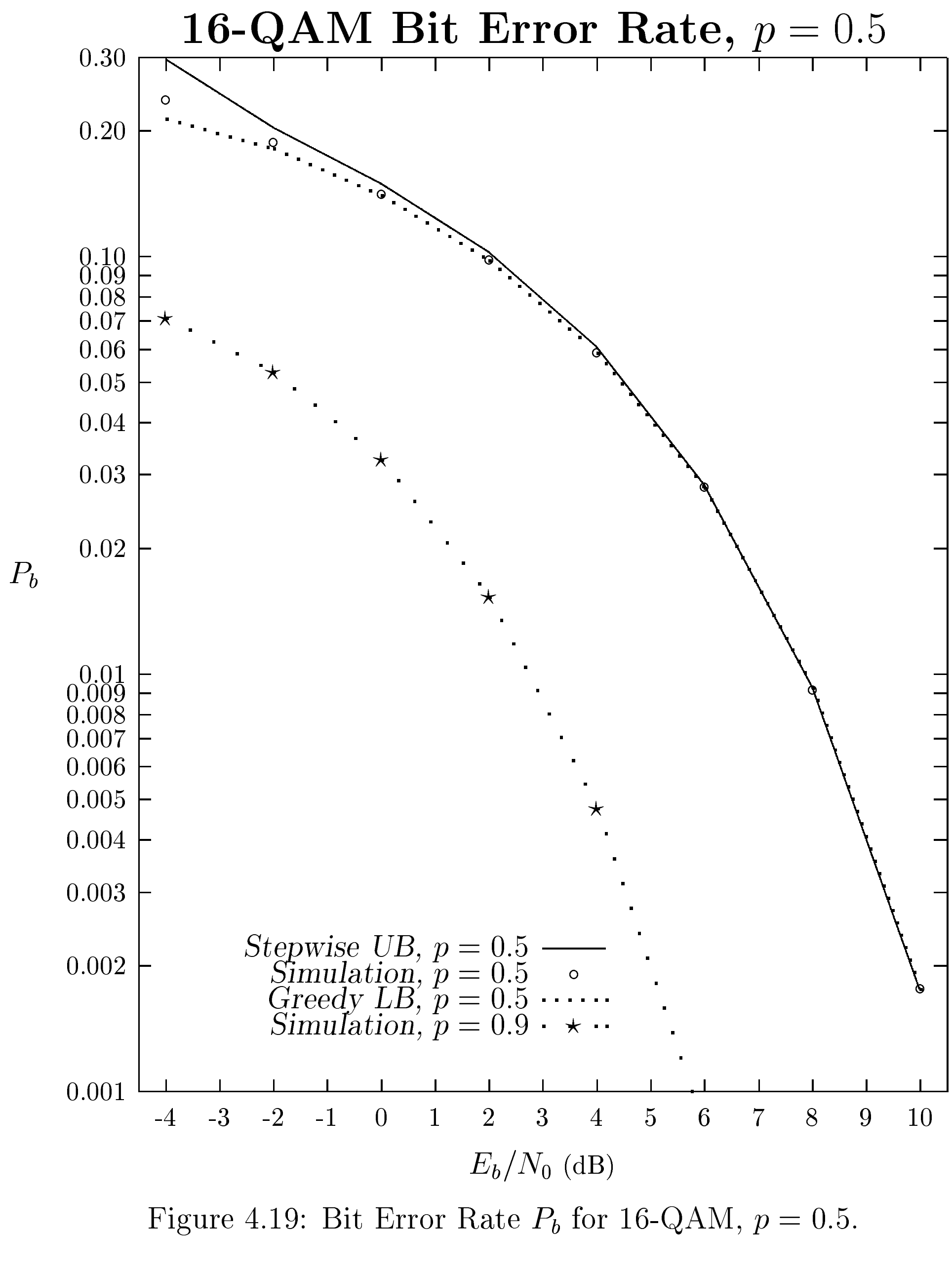

Figure 4.19: Bit Error Rate P_b for 16-QAM, $p = 0.5$.

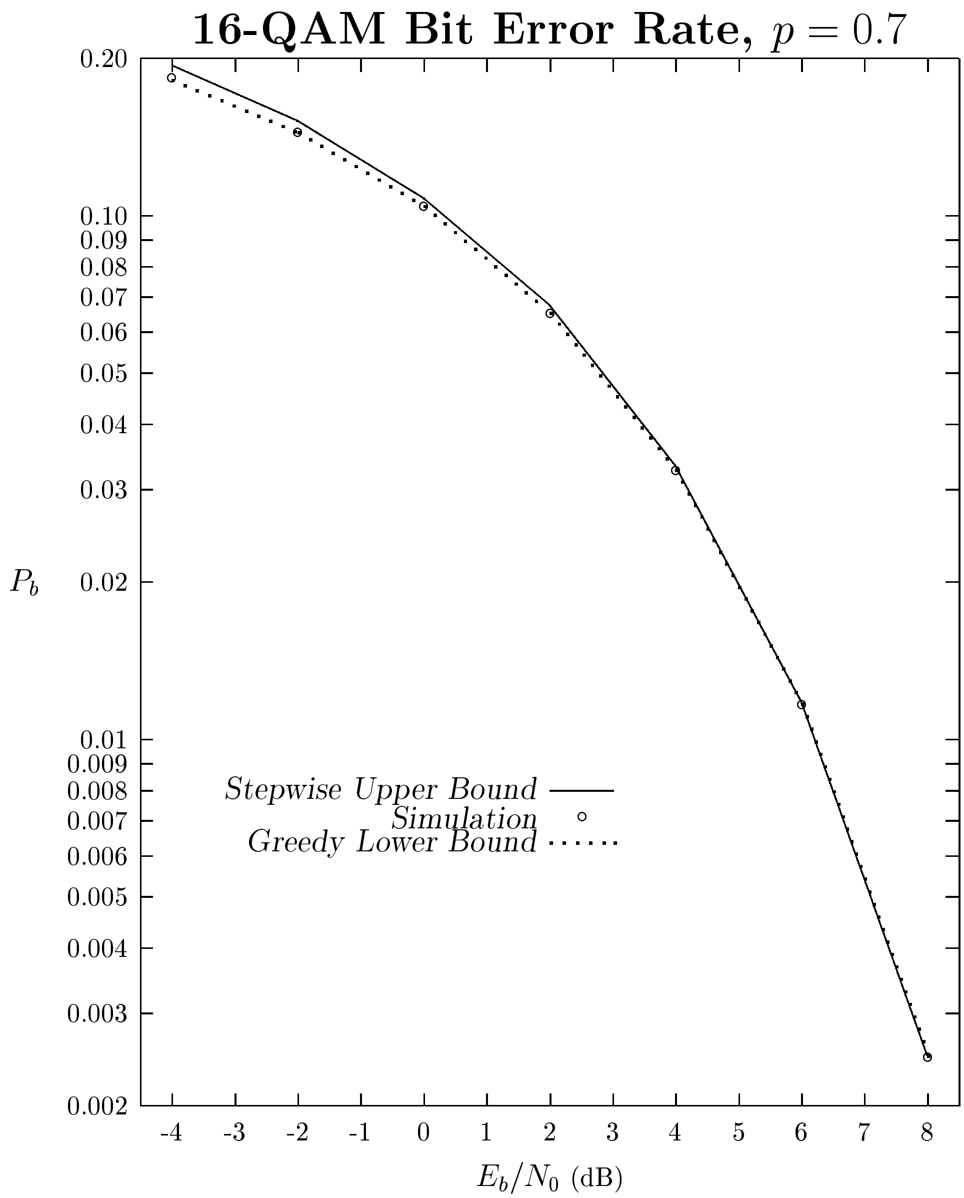

Figure 4.20: Bit Error Rate P_b for 16-QAM, $p = 0.7$.

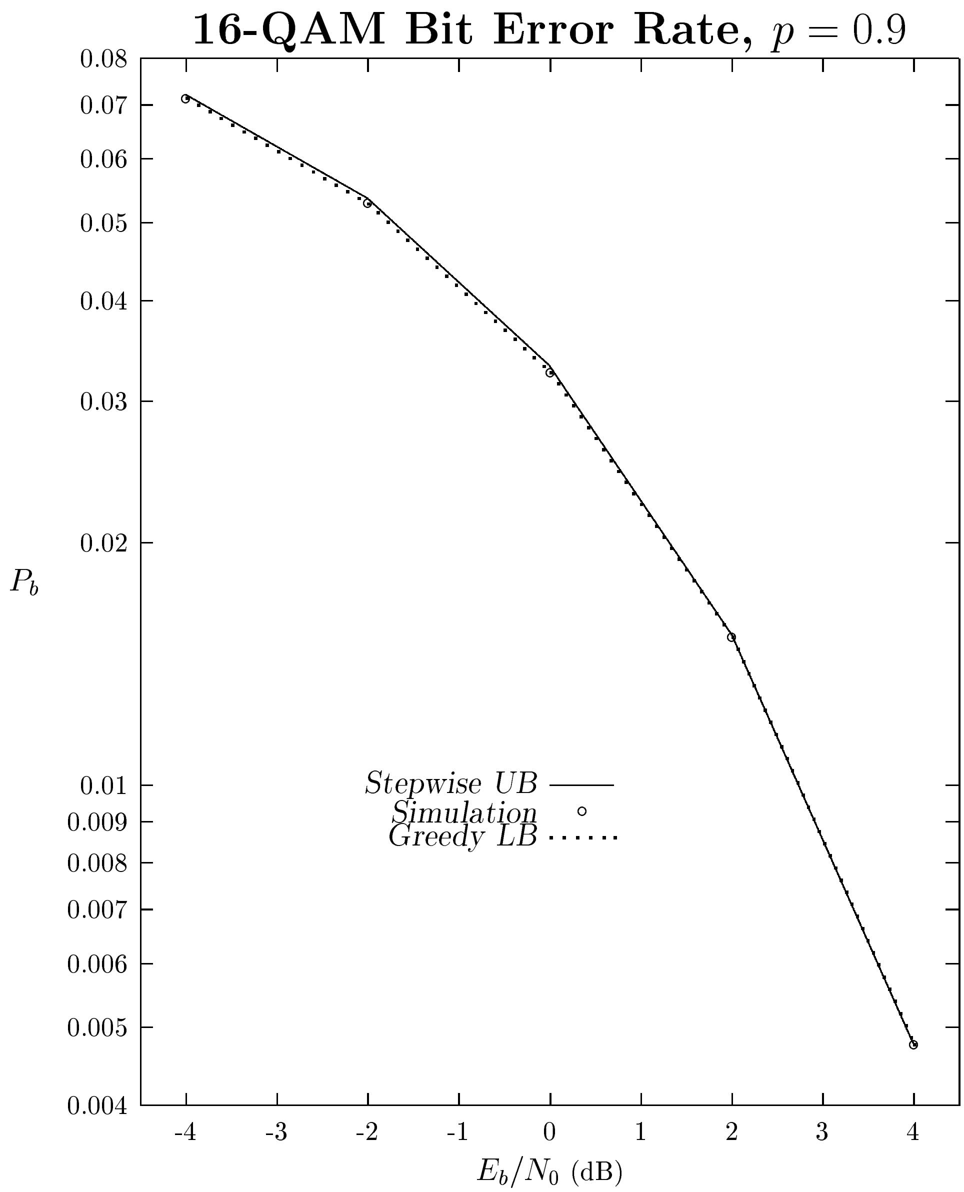

Figure 4.21: Bit Error Rate P_b for 16-QAM, $p = 0.9$.

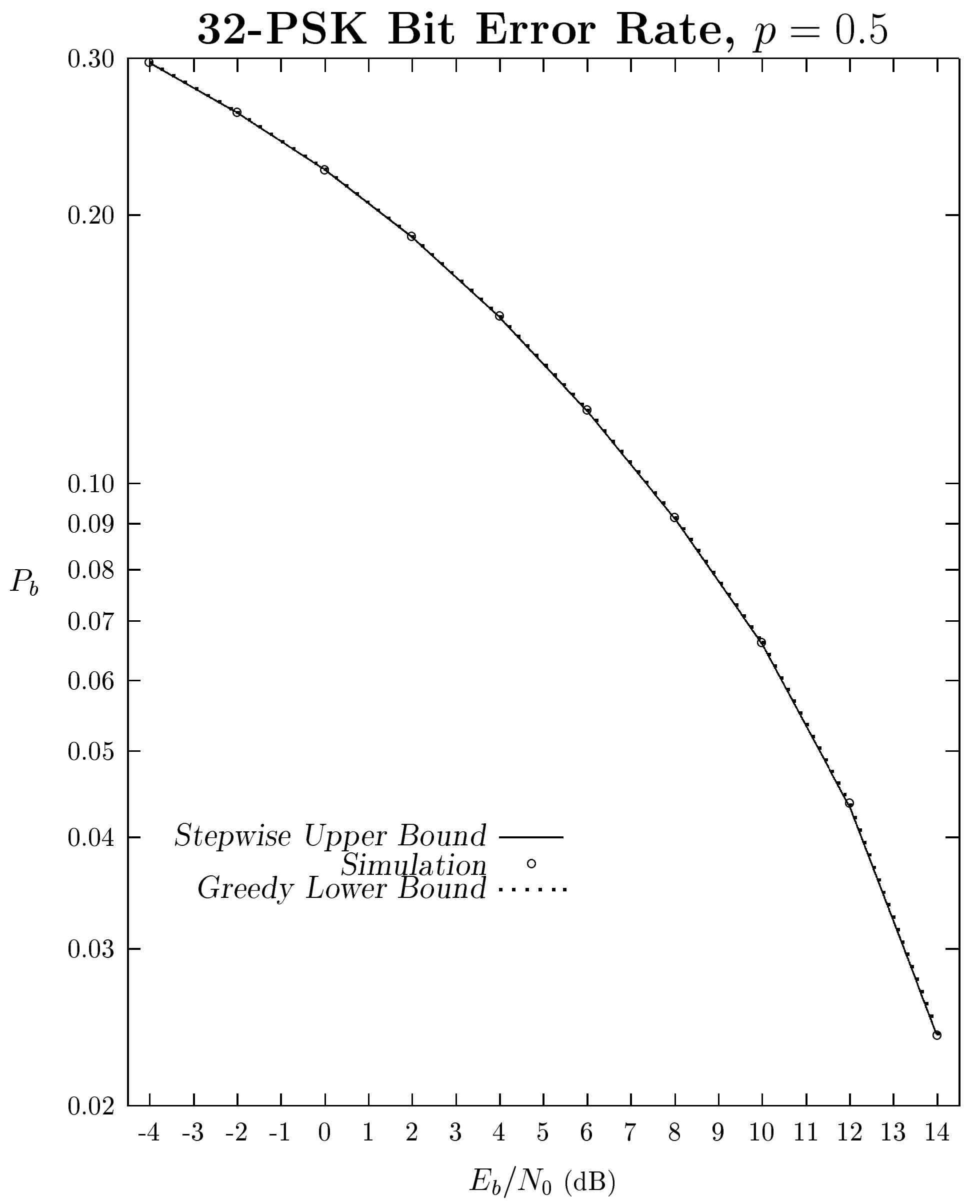

Figure 4.22: Bit Error Rate P_b for 32-PSK, $p = 0.5$.

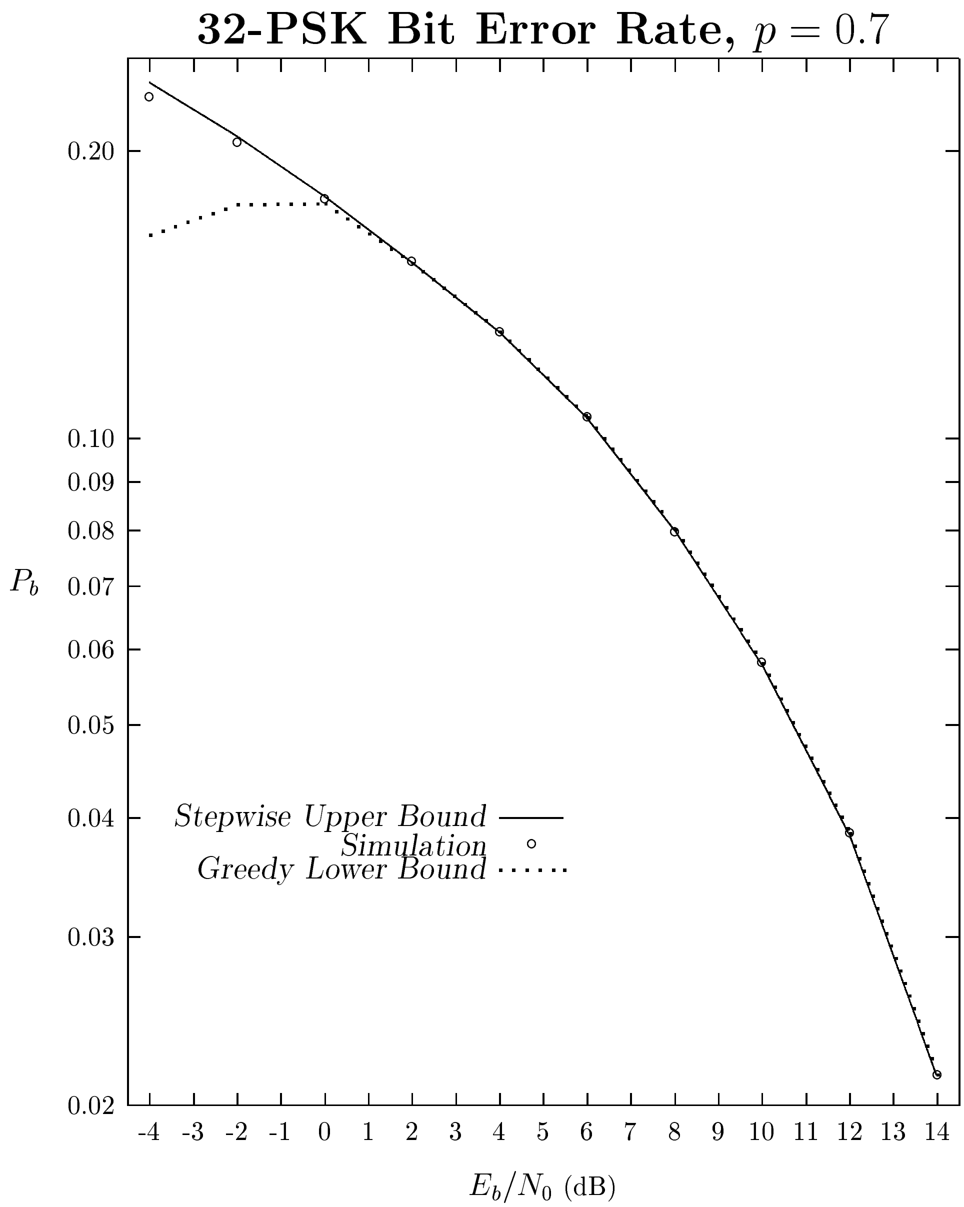

Figure 4.23: Bit Error Rate P_b for 32-PSK, $p = 0.7$.

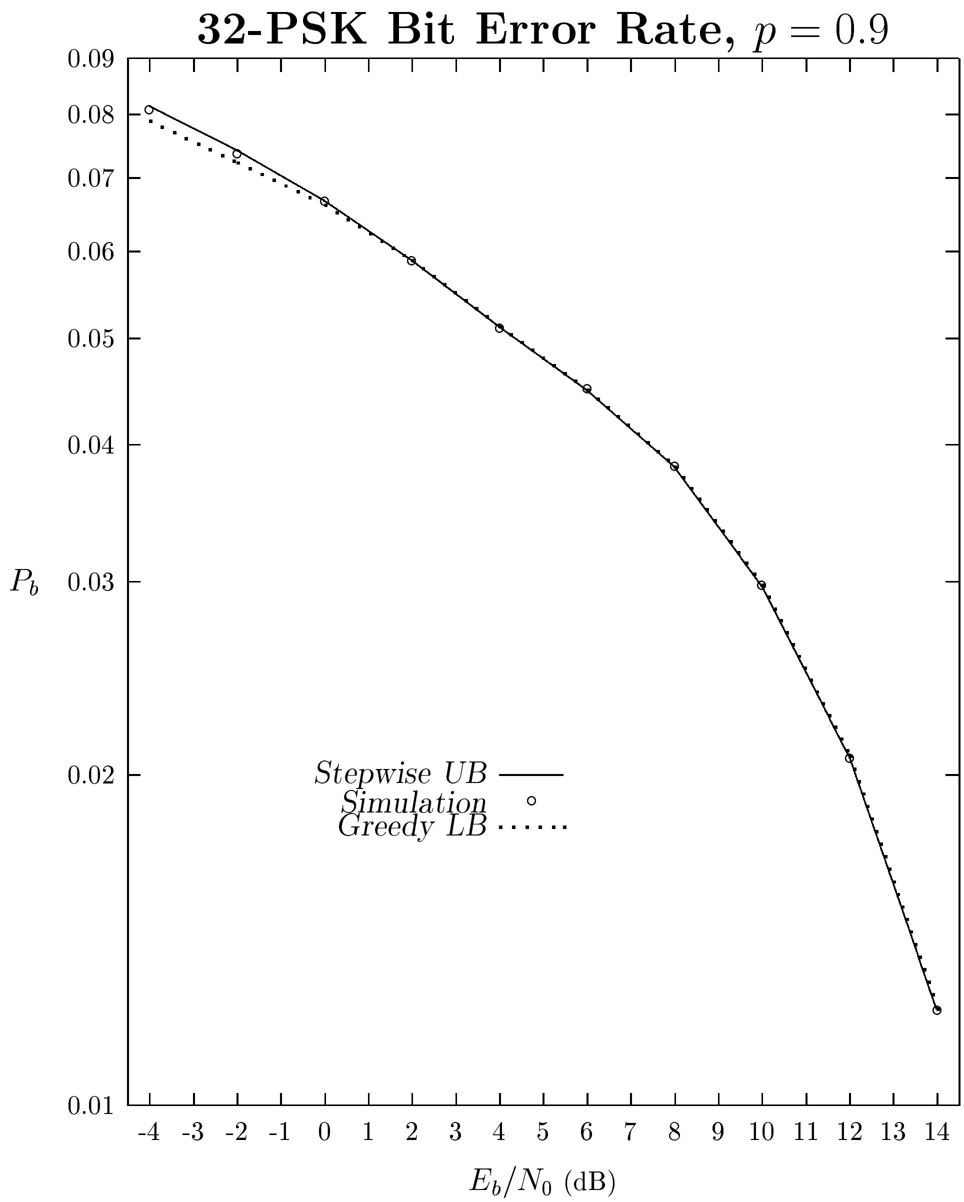

Figure 4.24: Bit Error Rate P_b for 32-PSK, $p = 0.9$.

Chapter 5

Conclusions and Future work

5.1 Summary

In this thesis, two lower bounds and one upper bound on the probability of a union of a finite family of events were derived and applied to estimate the error performance of two dimensional modulation techniques subject to an AWGN communication channel. These bounds are expressed in terms of the individual and pairwise event probabilities. Thus, when they are applied to communication systems, less complexity and reasonable efficiency can be achieved.

The KAT lower bound has proved to be at least as good as two similar lower bounds, one by de Caen (1996) and another by Dawson and Sankoff (1967). Actually, these two lower bounds are special cases of the KAT bound. If $\theta_i = 0$ for all i, the KAT bound reduces to de Caen's lower bound; if $\frac{\beta_i}{\alpha_i} = C$ for all i, where C is a constant, then $\theta_i = \theta$ for all i and the KAT lower bound reduces to the Dawson-Sankoff lower bound.

The Stepwise bound is an algorithmic bound. Although the Stepwise lower bound is a suboptimal instance of Kounias inequality, it is more useful in practice since its complexity increases quadratically (as opposed to exponentially) with N. This point is important especially when we consider coded signals in the future. In that case, N, the number of the signals, will be very large.

The application of these bounds to non-uniform signaling over the AWGN channel provided extremely good results for a wide range of channel conditions for p= 0.5, 0.7 and 0.9 in both SER and BER cases. The exact values or simulation results were also provided. The Stepwise bound and the Greedy algorithm bound coincided perfectly for uniform signals, even in very low SNR's, where the usual union bound is weak.

5.2 Future work

The work we have done is based on channel uncoded communication systems. It straightforwardly leads to some other directions.

- The application of these bounds to channel coded communication systems. In a previous work, Séguin [23] employed de Caen's lower bound to derive a lower bound on the probability of codeword error for M-ary signal derived from binary linear codes sent over the AWGN channel with the ML decoder. We will examine the application of the KAT, Stepwise and Greedy bounds to systems used in conjunction with Hamming code or Golay code first. Then, we will address the performance of Turbo codes.

- The application of these bounds to Rayleigh fading channels. In [4, 10, 25], Craig's method [7] is widely used and some interesting results are obtained. However, that work is still based on the ML decoding criterion. Since Craig's idea is also valuable for the MAP case, and no closed-form expression can be found in the literature for the SER and BER for fading channels, these bounds will be applicable to bound the SER and BER.

Appendix A

Digital Modulation

We provide some basic introduction on communication systems [21, chapter 4].

A.1 Digitally modulated signals

When we want to transmit digital information over a communication channel, a modulator is used to map the digital information into analog waveforms. The basic idea of the mapping is that taking blocks of $k = \log_2 M$ binary digits at a time from the information sequence $\{a_n\}$ (which we intend to transmit) and selecting one of $M = 2^k$ deterministic, finite energy analog waveforms $\{s_m(t), m = 1, 2, \ldots, M\}$ for transmission over the channel. These waveforms may differ in either amplitude or in phase or in frequency, corresponding to three basic modulation techniques: Amplitude-shift keying (ASK), phase-shift keying (PSK) and frequency-shift keying (FSK).

Phase-shift Keying (PSK) In digital phase modulation, the M signal waveforms are represented as

$$s_m(t) = g(t) \cos[2\pi f_c t + \frac{2\pi}{M}(m-1)],$$

$$m = 1, 2, \ldots, M, 0 \leq t \leq T$$

where $g(t)$ is the signal pulse shape and $\theta_m = \frac{2\pi}{M}(m-1)$ are the M possible phases of the carrier that convey the transmitted information.

The signal waveforms may be represented as a linear combination of two orthonormal signal waveforms, $f_1(t)$ and $f_2(t)$, i.e.,

$$s_m(t) = s_{m1}f_1(t) + s_{m2}f_2(t),$$

where

$$f_1(t) = \sqrt{\frac{2}{\varepsilon}}g(t)\cos 2\pi f_c t,$$

$$f_2(t) = -\sqrt{\frac{2}{\varepsilon}}g(t)\sin 2\pi f_c t$$

and

$$\varepsilon = \int_0^T g^2(t)dt.$$

and the two-dimensional vectors $\mathbf{s_m} = [s_{m1}, s_{m2}]$ are given by

$$\mathbf{s_m} = [\sqrt{\frac{\varepsilon}{2}}\cos\frac{2\pi}{M}(m-1), \sqrt{\frac{\varepsilon}{2}}\sin\frac{2\pi}{M}(m-1)]$$

$$m = 1, 2, \ldots, M.$$

Quadrature Amplitude Modulation (QAM) The signal waveforms for QAM may be expressed as

$$s_m(t) = A_{mc}g(t)\cos 2\pi f_c t - A_{ms}g(t)\sin 2\pi f_c t,$$

$$A_{mc} = (2i - 1 - \sqrt{M})\frac{d}{2},$$

$$A_{ms} = (2j - 1 - \sqrt{M})\frac{d}{2},$$

$$m = 1, 2, \ldots, M, 0 \leq t \leq T$$

$$i, j = 1, 2, \ldots, \sqrt{M}.$$

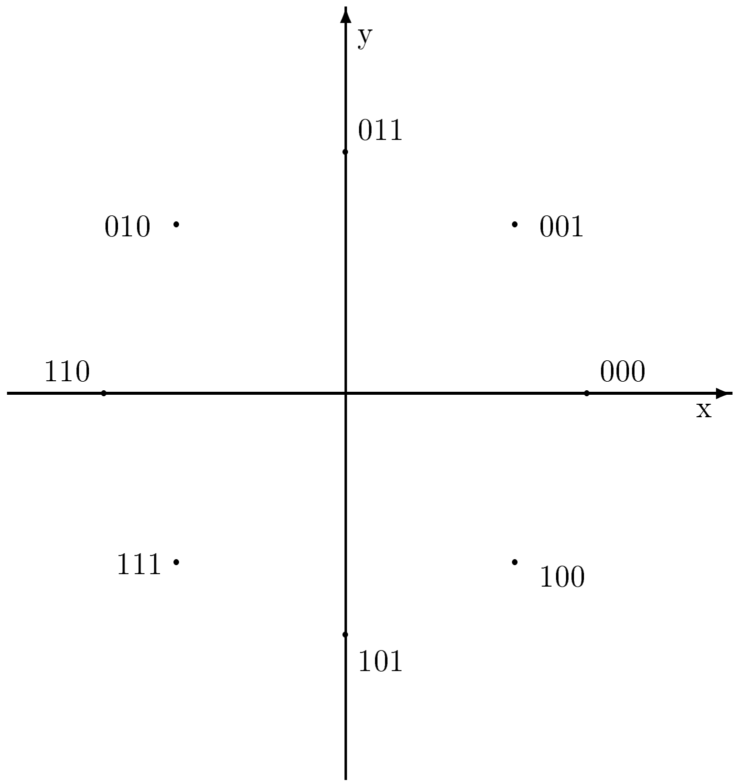

Figure A.1: 8-PSK Constellation with Gray Mapping.

where d is the distance between adjacent signals. It is not necessary for A_{mc} to be equal to A_{ms} and $g(t)$ is the signal pulse. The corresponding two-dimensional vectors are

$$\mathbf{s_m} = [A_{mc}\sqrt{\frac{\varepsilon}{2}}, A_{ms}\sqrt{\frac{\varepsilon}{2}}]$$
$$m = 1, 2, \ldots, M.$$

Gray Code Mapping Gray coding is a mapping with the property that binary codewords for every pair of adjacent signals differ in only one position. This mapping is preferrable for memoryless channels because the most likely demodulation errors caused by the channel noise result in the selection of a signal adjacent to the transmitted one. In this case, only a single-bit error occurs in the binary codeword. Figures A.1 and A.2 illustrate Gray mapping for 8-PSK and 16-QAM constellations.

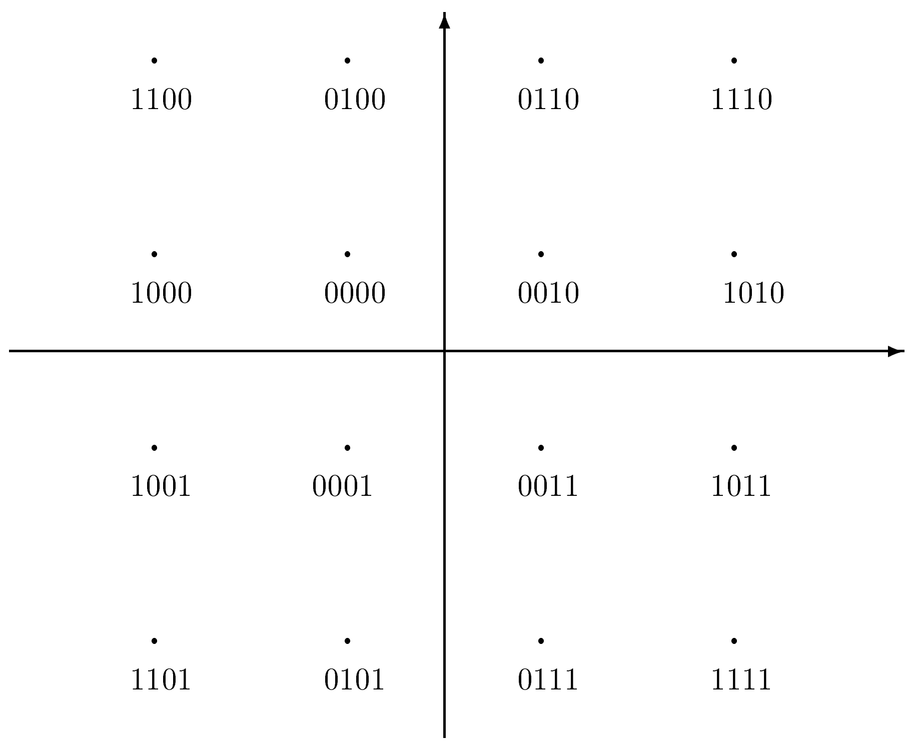

Figure A.2: 16-QAM Constellation with Gray Mapping.

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